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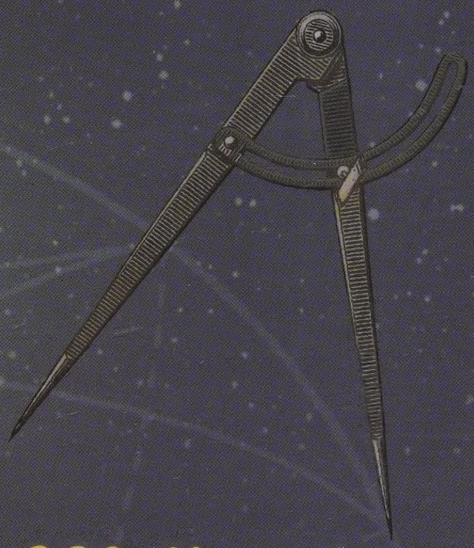
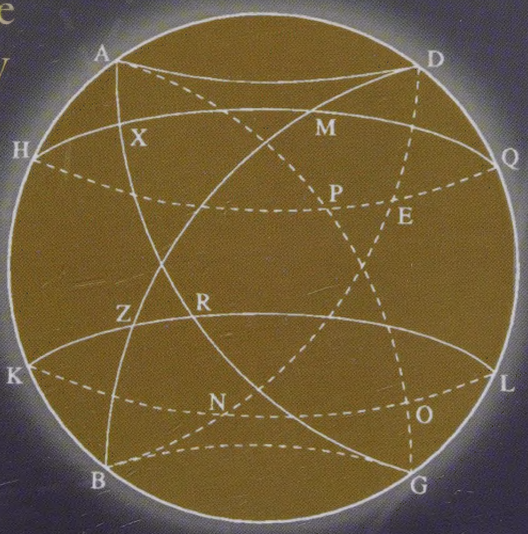


Euclid's *Phaenomena*

A Translation and Study of a Hellenistic Treatise in Spherical Astronomy

J. L. Berggren
R. S. D. Thomas



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A Translation and Study of a Hellenistic Treatise in Spherical Astronomy

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Preface to the Second Printing

This work was first printed in 1996 by Garland Publishing as No. 4 in the series *Sources and Studies in the History and Philosophy of Classical Science* (Alan C. Bowen and Francesca Rochberg, eds.). The printed copies sold out and the work has been out of print for well over five years. Since there seemed to be a steady interest in obtaining copies of the work, the authors approached the American Mathematical Society and were pleased to learn that they, with the London Mathematical Society, were interested in republishing it in the AMS/LMS *History of Mathematics* series. Both authors take this opportunity to thank both societies and AMS Acquisitions Editor Edward Dunne for their cooperation in making this work available again.

Euclid's *Phaenomena* belongs to a branch of ancient mathematical sciences that the Greeks called spherics *(ta sphairika)*, whose purpose was to explain the geometry of the celestial sphere and its circles as well as the consequences of that geometry for observers on a spherical earth located at the center of the celestial sphere. Ancient writers whose works were part of this genre included, in addition to Euclid, Autolycos, Aristarchos, and Theodosios.

Some recent works relevant to this area, not cited in the previous printing of the *Phaenomena*, are:

Czinczenheim, Claire. *Edition, traduction et commentaire des "Sphériques" de Théodose.* Thesis, Université de Paris IV (2000).

Berggren, J. L., and R. S. D. Thomas. "Mathematical Astronomy in the Fourth Century B. C. as Found in Euclid's Phaenomena." *Physis.* (N.S.) Vol. XXIX (1992), pp. 7–33.

Evans, J. C., and J. L. Berggren. *Geminos: Introduction to the Phenomena.* Princeton University Press (to appear 2006).

Evans, J. C. *The History and Practice of Ancient Astronomy* (especially Chapters 2 & 3). New York : Oxford University Press (1998).

Thomas, R. S. D. "Euclid's Non-euclidean Geometry" in *Daimonopylai* (R. B. Egan and M. A. Joyal, eds.). Winnipeg: University of Manitoba Centre for Hellenic Civilization (2004), pp. 469–478.

Lorch, R. "The Transmission of Theodosius' *Sphaerica*", in: *Mathematische Probleme im Mittelalter: Der lateinische und arabische Sprachbereich* (Menso Folkerts, ed.). Wiesbaden: Harrassowitz Verlag (1996), pp. 159–183.

The following errors/misprints in the previous printing should be noted:

p. 15, f.n. 22: C. Czinczenheim informs us in a letter that the citation referred to occurs in Theodosios's *On Habitations*, rather than in his *Sphærica.*

p. 28: Delete the comma following 'earth' in line 9.

p. 118 (s.v. "Similar arcs"): Change "equal angels" to "equal angles."

p. 122: Change "Bruin, E." to "Bruin, F.".

PREFACE

It has often been remarked how completely the masterpieces of Hellenistic exact sciences—for example, Euclid's *Elements*, Apollonius' *Conics*, and Ptolemy's *Almagest*—superseded their predecessors, so that we know almost nothing of, say, Hippocrates of Chios' *Elements*, Euclid's *Conics*, and Apollonius' astronomical writings. As a work of the fourth century BC, Euclid's *Phaenomena* is, therefore, a rare testimony to some of the problems and methods of Greek astronomy before Apollonius, to say nothing of Hipparchus and Ptolemy. Its very survival indicates that its many copyists—from Hellenistic antiquity through the Islamic and Latin Middle Ages and up to the Renaissance—thought that, despite the great advances in astronomy since Euclid's time, the text retained its value as an introduction to the geometry of circular arcs on the sphere and to the problem of rising times of ecliptic arcs.

Yet, in spite of the light the *Phaenomena* sheds on early Greek mathematical astronomy and its record of continuous study—in the East and the West—from the time of Euclid to that of Copernicus, the work has been underrated. Heath does not even summarize it, however briefly, in his *History of Greek Mathematics*; and the lack of a modern translation is surely symptomatic of present views of the work. We feel that much of the modern attitude towards the work reflects a rejection of those parts of Greek science that do not, or cannot be made to, come up to modern expectations of what science should be.

Our hope is that by making the *Phaenomena* available in English we will encourage its study not only by historians of science who have no Greek, but its inclusion in courses in the history of science from which it has up to now been excluded. How it happened that we decided to make Euclid's *Phaenomena* available is a story which is quickly told.

In 1986, one of us was the local organizer for the annual meeting of the Canadian Society for History and Philosophy of Mathematics / Société canadienne d'histoire et de philosophie des mathématiques in Winnipeg.

At that meeting the other of us presented a paper on the period of Greek astronomy into which *Phaenomena* falls and discussed the contents both of that treatise and those by Autolycus. In answer to a question from the audience at the end of the talk, he said that, unlike Autolycus' treatises, the text of Euclid's work was not available in English or French. Thinking that this was remarkable, regrettable and remediable, the other of us wrote to suggest that publication of an English translation with commentary would be a worthwhile joint project. We agreed and then began nearly a year later, having checked with the editor of this series, Alan Bowen, that no one else was working on such a translation. The translation was completed in a year or two; and the subsequent time has been spent preparing the other parts of this volume including the figures, the writing and revision of a paper[1] based on it, and final revisions of the translation, as well as preparing electronic files.

We thank Karl Berggren for preparing an initial version of the diagrams based on those in Menge's text. We also thank the following individuals who have, by their comments and criticisms, aided in improving both our translation and our commentary in content and style: Alan Bowen, Bernard Goldstein, Alexander Jones, and Wilbur Knorr. We alone, of course, are responsible for any errors that remain.

J.L.B. thanks both Harvard University's Department of the History of Science (where he was Visiting Scholar) and Quincy House (where he and his wife were Resident Affiliates) for their generous hospitality extended during a sabbatical leave in 1990–91 and a research term in the Fall of 1992. R.S.D.T. thanks the Department of Mathematics and Statistics at Simon Fraser University for its hospitality during a sabbatical leave in 1988–89. We also thank the Hill Monastic Manuscript Library for making available to us a microfilm of cod. Vind. gr. 31.13, and the Vatican Library for a microfilm of cod. Vat. gr. 204.

Finally, our greatest debt is to our wives, whose love and encouragement has been both support and stimulus in this endeavor, as in so many others. We dedicate this book to them as a token of our gratitude.

---

[1] J. L. Berggren and R. S. D. Thomas. 1992. 'Mathematical Astronomy in the Fourth Century BC as Found in Euclid's *Phaenomena*' *Physis: Riv. Internaz. Storia Sci.* (N.S.) 29:7–33.

Euclid's *Phaenomena*

If we mean, then, to turn the soul's native intelligence to its proper use by a genuine study of astronomy, we shall proceed, as we do in geometry, by means of problems, and leave the starry heavens alone.

Plato

1

Introduction

The purpose and strategy of the *Phaenomena*

The *Phaenomena* is a geometrical treatment of some fundamental problems related to the risings and settings of stars and of important circular arcs on the celestial sphere. In fact, just over half its theorems (the last ten) are devoted to one of these problems, that of determining the length of daylight on a given day at a given locality, the two data on which the length of daylight obviously depends. Euclid's is the earliest extant treatise dealing with this particular question. Theodosius (late second century BC) and Menelaus (end of the first century AD) also wrote treatises bearing on this problem: indeed, Neugebauer has described its solution as 'one of the major goals' of their spherical geometry.[1] Some fifty years after Menelaus completed his *Sphaerica*, Ptolemy wrote his *Syntaxis mathematica* or *Almagestum* (*Almagest*); and the earliest record we have of an exact, geometrical solution to the problem of finding the length of daylight is in that work [cf. *Alm.* ii 9].

There were, however, earlier solutions. The oldest known method for solving this problem is found in a Ramesside papyrus of the twelfth century BC [Neugebauer 1975, 706], which contains approximate values for the lengths of daylight in each of the twelve months of the Egyptian year calculated on the basis of direct linear interpolation between a maximum value of 18 hours and a minimum of 6 hours. Babylonian solutions based on the same idea, but with different values for the maximum and minimum lengths of daylight, appear much later, an early specimen being contained

[1] The historical details given here and in the following paragraph may be found in Neugebauer 1975, 706–733.

in a tablet dating from about 400 BC [Neugebauer 1975, 709], still well before Euclid.

At some time, however, Babylonian astronomers devised a new method based on the insight that the length of daylight on a given day is the length of time[2] it takes a certain arc of the ecliptic to rise over the horizon. More specifically, they realized that the length of daylight is the time it takes the semicircle of the ecliptic following the Sun to rise as the point on it occupied by the Sun moves across the sky from the eastern to the western horizon. This insight that the problem of finding the length of daylight could be solved by calculating the 'rising times' of ecliptic arcs led to an indirect solution according to which one determines the length of daylight in two steps:

(1) by assigning rising times, through calculation or the use of a simple linear scheme, to a set of consecutive arcs covering the ecliptic, say the individual signs, thirds of signs or even individual degrees, and then

(2) by computing the length of daylight on a given day as the sum of the rising times for the set of these arcs spanning the particular 180° of ecliptic arc rising after the Sun on the given day.[3]

Schemes of this type, which are intermediate between the earlier arithmetic solutions and the elegant trigonometric solutions of Ptolemy, rely—implicitly or explicitly—on symmetries which Ptolemy later proves, in order to obtain the rising times used in (1) above. Unlike Ptolemy's solutions, however, such schemes employ not trigonometrical but arithmetical methods. These schemes were known to Greek writers of the Hellenistic period, as the *Anaphoricus* (*On Risings*) by Hypsicles of Alexandria, who probably wrote before 150 BC, attests. In fact, it is our view that Euclid knew of these older arithmetic approaches to the problem of determining rising times, and that one of his goals in writing the *Phaenomena* was to demonstrate geometrically the assumptions behind this arithmetic method.

---

[2] Babylonian texts express such times in units of *uš*, each of which was $1/360$ of a twenty-four hour period, that is, four minutes. Since 360° of the equator also rise at a uniform rate over the horizon in the same period, an *uš* is, in fact, the time it takes any one-degree arc of the equator to rise. For this reason they were called time degrees by the Greeks [see Neugebauer 1975, 367] and were used by Ptolemy in his table of rising times.

[3] Unfortunately, the two texts in which such schemes occur (200 and 200b in Neugebauer 1955, i 87 and 214) are not dated, but they are part of a Babylonian archive of which the bulk of the material is datable to the middle of the second century BC [see 1955, i 9–11]. However, as Neugebauer emphasizes, it is futile to try to date the method from the text, which can only give a latest possible date.

Euclid organizes his exposition as follows:[4]

Introduction:

1. Arguments for the sphericity of the cosmos and its uniform rotation about an axis. (The sphericity of the Earth is tacitly assumed.)
2. Definition of the principal celestial circles of interest to astronomers: meridian, tropic circles, arctic circle, horizon, equator, and ecliptic, together with a demonstration that the last three are great circles.

Preliminary topics:

3. Argument that the Earth is at the centre of the cosmos (Proposition 1).
4. Determination of when the ecliptic and meridian or ecliptic and horizon will be mutually perpendicular (Proposition 2).

On the risings and settings of stars:

5. A fixed star that rises and sets does so always at the same point on the horizon (Proposition 3).
6. Comparative order of risings and settings of stars located on great circles intersecting or not intersecting the arctic circle (Propositions 4 and 5).
7. Diametrically opposite stars on the ecliptic or equator rise and set in tandem (Proposition 6).

On ecliptic arcs, where they rise and how long they take to rise:

8. The arcs of the horizon where the whole ecliptic or individual signs rise (Propositions 7 and 8).
9. Comparison of the rising times of two given semicircles of the ecliptic (Proposition 9), and the following is either stated or trivially deducible:

   > Daylight is longest when the Sun is at the beginning of Cancer; shortest when it is at the beginning of Capricorn. For any other sign in Table 1, the length of daylight when the Sun is at its beginning is less than it is when the Sun is at the beginning of the signs above it. Moreover, the length of daylight is the same when the Sun is at the beginning of signs side by side in this table.[5]

   Next, Proposition 10 allows one to pass from a comparison of the rising times of semicircles to a comparison of the rising times of smaller arcs.
10. Demonstration that the rising (setting) time of one arc is equal to the setting (rising) time of the other if either

[4] See English Glossary for definitions of technical terms.

[5] See Figure 1 for a diagram of the ecliptic as viewed from its north pole. On the ecliptic, see also pp. 29–30 below.

	Cancer
Leo	Gemini
Virgo	Taurus
Libra	Aries
Scorpio	Pisces
Sagittarius	Aquarius
	Capricorn

Table 1.

(a) the two arcs are opposite with respect to the centre of the ecliptic, or

(b) if they are equidistant from a tropic (Proposition 11 and the lemma following Proposition 13).

Proposition 12 then compares the setting times of signs in the semicircle following Cancer,[6] and Proposition 13 does the same for rising times in the semicircle following Capricorn.[7] Propositions 12 and 13 taken together also assert the equality both of rising and setting times for arcs symmetrically situated with respect to the equator. Thus, if one places signs with equal rising and setting times opposite each other in a table one gets:

Capricorn	Gemini
Aquarius	Taurus
Pisces	Aries
Sagittarius	Cancer
Scorpio	Leo
Libra	Virgo

Table 2.

Figure 1, in which vertical lines join signs with equal rising times, and horizontal lines join the beginnings of semicircles with equal rising times, summarizes the information Euclid provides on the problem of rising times.

Conclusion:

11. Propositions 14–18 compare the times it takes equal arcs of the ecliptic to cross the visible and invisible hemispheres. Since the Sun is not a mathematical point but occupies about 1/2° of the ecliptic, and also

[6] Since the Sun enters Cancer at the summer solstice, the rising times of these signs add up to the length of daylight on the longest day of the year.

[7] The Sun enters Capricorn on the shortest day of the year, the winter solstice.

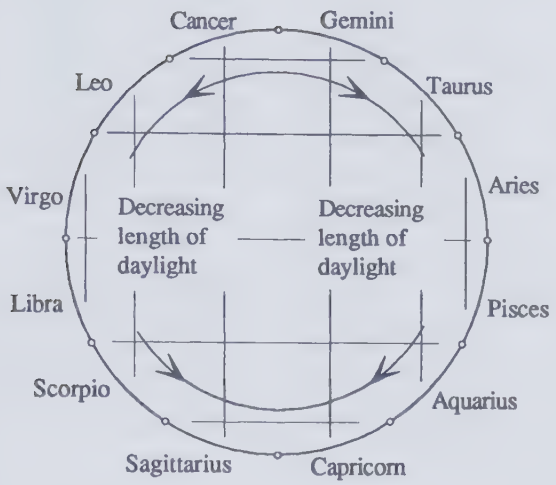

Figure 1. Ecliptic with symmetries of rising times shown.

moves during the course of a daytime about 1/2° (on the average) along the ecliptic (counterclockwise in Figure 1), it follows that the length of daylight is precisely equal to the time it takes that arc of the ecliptic occupied by any part of the solar disk during the course of daylight to cross the visible hemisphere.

One observes in this brief account that, beyond stating the equality of rising or setting times for arcs symmetrically situated with respect to the equator, Euclid does not compare rising times of arcs in the semicircle following Cancer nor (correspondingly) setting times in the semicircle following Capricorn. The reason is that no simple inequalities, like those for the cases he does discuss, apply here. To see this, consider *Alm.* ii 8 in which Ptolemy gives the following values for the rising times of the signs of Cancer, Leo, and Virgo,[8] as computed for various localities that are classified according to maximum hours of daylight [see Table 3]. Thus, for localities near the equator, Cancer is the sign that takes longest to rise; for more northerly localities, it is Leo.

As Pappus of Alexandria (*fl.* ca. 300 AD) pointed out in his commentary on the *Phaenomena*, Euclid could not have stated inequalities corre-

[8] The table indicates *inter alia* that for a locality in which the maximum length of daylight is $14\,1/2^{h}$ (in Rhodes, for example), an arc of the equator measuring $36°\ 28'$ rises while Virgo does. Since $1° = 4$ minutes of rising time, this may be understood to mean that Virgo takes $2^{h}\ 25'\ 52''$ to rise above the horizon.

Hours	Cancer	Leo	Virgo
12	32°16′	29°54′	27°50′
12½	32°51′	31°20′	29°34′
13	33°26′	32°44′	31°20
14½	35°15′	37°02′	36°28′
16	37°15′	41°25′	41°20′
17	38°47′	44°22′	44°21′

Table 3.

sponding to Proposition 13 for signs in the semicircle following Capricorn [cf. *Coll.* vi 108 *et seq.*]. According to Pappus [*Coll.* vi 55], Hipparchus (*fl. ca.* 150 BC) was, in fact, the first to show this by numerical examples; but it was Menelaus who finally solved the problem left open by Euclid's *Phaenomena*, in his own treatise on the setting times of arcs in semicircles of the ecliptic following Capricorn.

Pre-Euclidean works on the subject of the *Phaenomena*

A reading of the *Phaenomena* will demonstrate convincingly what the above list of topics suggests, namely, that Euclid was not the earliest writer on these topics. The sophistication of many of the propositions that Euclid assumes about the geometry of circles on the sphere [see pp. 19–27, below] indicates that he had a text or texts on that subject which we, following the Greeks, shall call spherics.

This observation, however, introduces two puzzles. The first is that, since the sole motivation for the abstract, geometrical propositions Euclid assumes is, apparently, their use in astronomy, there must have been, before Euclid's time, a development not only of a geometrical theory of spherics but also of its use in the study of the astronomical problems motivating it. There is evidence of this pre-Euclidean geometrical theory of spherics in Euclid's tacit assumptions and in Theodosius' *Sphaerica*; but no trace remains of its earlier use in astronomy. The treatises by Autolycus, *De sphaera quae movetur* (*On a Moving Sphere*) and *De ortibus et occasibus* (*On Risings and Settings*),[9] do not solve this puzzle but only emphasize it, since they too use some of the same geometrical theorems without any comment. Since, unlike Euclid's *floruit*, which is traditionally taken to be around 300 BC, Autolycus' *floruit* can be assigned with some assurance to

[9] On the goal and content of the former treatise (hereafter, *De sphaera*), see Berggren 1991.

333–300 BC [see Mogenet 1950, 5–9], one has an approximate latest date for the development of the requisite geometrical theory of the sphere.

This puzzle, however, turns out to be more apparent than real if one assumes that Euclid's *Phaenomena* superseded the earlier treatments of the basic phenomena of the celestial sphere. This would parallel the case of his *Elements*, another work whose success meant the disappearance of earlier works on the subject. This seems to have been a common fate of early scientific works; indeed, it appears that Euclid's *Conics* suffered the same fate when that of Apollonius appeared. So there is nothing at all unusual about certain strata of ancient works disappearing without a trace; in fact, one often feels that what has to be explained is not the disappearance of ancient works but their preservation.[10]

The other puzzle, however, is real: one must face the question of who was responsible for the pre-Euclidean mathematical astronomy. A favorite candidate has been Eudoxus of Cnidus, a contemporary of Plato, whose contributions to mathematics were mentioned by Archimedes. In fact, among Eudoxus' works on astronomy was one with the promising title *Phaenomena* (the first reported work with that title) and another called the *Enoptron*.[11] According to Hipparchus, the two works agreed 'very much with each other in practically all things' [cf. Manitius 1894, 8.15–18].

However, when one examines the available evidence concerning the contents of these two books for material resembling that in Euclid's *Phaenomena*, one finds very little. Since Hipparchus, in the tradition of Hellenistic scholarship, quotes Eudoxus primarily to disagree with him, it is difficult to get a conception of Eudoxus' work as a whole. It is clear, however, that these books contained an extensive description of the constellations on the celestial sphere and their placement relative to each other, with descriptions of this sort:

> Between the Ursae lies the tail of Draco, having its outermost star over the head of Ursa Major. It curves by the head of Ursa Minor and then stretches under its feet. Making another turn here again it rears its head up and holds it forward. [Manitius 1894, 8.22–10.4]

Eudoxus also describes the circle of the winter tropic in these words:

> On it are the middle parts of Aries, the feet of Aquarius... , the mast of Argus, the back and chest of Centaurus, Lupus and the

[10] The loss of Eudemus' writings, among which was, apparently, a *Historia astrologiae* (*History of Astronomy*), is to be regretted.

[11] An ἔνοπτρον is something one looks into, such as a mirror; the word may thus signify a work that was an image in prose of the cosmos.

> stinger of Scorpio. Then it [the circle] joins up to the middle of Aries through Sagittarius. [Manitius 1894, 22.1–9]

Admittedly there are in Eudoxus' astronomical writings statements about circles on the celestial sphere of a more mathematical character. Thus, in his *Enoptron*, Eudoxus asserts that the tropic circles are cut by the horizon (of his locality) in a ratio of 5:3 [Manitius 1894, 22.19–22], and in his *Phaenomena* he gives the slightly larger ratio 12:7 [Manitius 1894, 28.8–13]. More importantly, Simplicius (sixth century AD), in his commentary on Aristotle, *Meta.* Λ 8, ascribes to Eudoxus a work called *De celeritatibus* (*On Speeds*) containing one of the earliest attempts to explain the principal phenomena of planetary motion using nested concentric spheres each rotating on an axis whose poles are carried on the next sphere out [cf. Heath 1913, 194]. The title and available evidence of the contents of this treatise, however, argue against its having contained the sort of considerations we find in Euclid.

There is, in conclusion, no evidence that Eudoxus wrote either on spherics or its astronomical application; nor are there any real clues as to when or by whom the theory was developed. We have argued elsewhere [cf. Berggren 1991] in support of an argument in Goldstein and Bowen 1983, that the development took place in the interval between the lifetimes of Eudoxus and Autolycus, but more than that we cannot say.

The integrity and authenticity of the *Phaenomena*

Two serious questions that must be addressed are whether Euclid wrote the *Phaenomena* and, if he did, whether what he wrote included the introduction. As for the first question, the ancient commentators strongly support the opinion that Euclid did write this work; and from the time of Galen onward they give details about the contents which make it certain that the work being referred to is in fact the present one. The earliest secure ancient testimony to the *Phaenomena* occurs in the writings of Galen, who was born probably around 130 AD, and wrote in his *De placitis Hippocratis et Platonis* (*On the Doctrines of Hippocrates and Plato*):

> Euclid showed in Theorem 1 of the *Phaenomena* in a few words that the Earth is in the midst of the cosmos, as a point or a centre, and the students trust the proof as if it were two and two is four. [cf. Müller 1874, 655.8]

The next references are found in Pappus' *Coll.* vi, which contains not only an extensive collection of lemmas to Euclid's *Phaenomena* but also

references to Propositions 2, 11, 12 and 13. During the sixth century, Marinus of Neapolis, in his commentary on Euclid's *Data*, refers to Euclid as the author of the *Phaenomena*; and John Philoponus in his commentary on Aristotle's *Physics* remarks that Euclid's *Phaenomena* is more 'physical' than Autolycus' astronomical writings because Euclid mentions not only 'motion' but 'substance' as well, by which he must mean Euclid's references to the Earth and the stars. Finally, there are the anonymous (and undated) scholiasts to Theodosius' *De diebus et noctibus* (*Days and Nights*), who refer to Proposition 16 of the *Phaenomena*, as well as the scholiasts to Autolycus' *De ortibus et occasibus*,[12] who mention Propositions 13 and 14 of the *Phaenomena*.

Given the tight mathematical structure of the work, the ancient references to specific propositions tend to support the authenticity of the other propositions that either furnish the prerequisites for the arguments in the propositions referred to in the *testimonia* or complete the theory that these propositions address. Indeed the work forms a logical whole, whose coherence is evident on reading it: Euclid starts with basic matters that are easily demonstrated, and progresses in a connected way to rather special topics requiring more sophisticated proof.

But what of the style and structure of the work? Are these such as to give us confidence that the work was written by Euclid? That the structure of the proofs differs only in one small respect from that of the proofs in the *Elements*,[13] may not count for much since it is difficult to be positive about how much the style and structure of either work owes to later editors; and there is no reason to assume that Euclid wrote all his works, which cover such a wide range of topics, in a single style. And it seems, at the very least, that there is no reason to deny that the work is by Euclid on the grounds of structure in the main body of the treatise.

However, when one considers the larger framework of the work some doubts arise. From a modern point of view, rigorous proofs depend on clearly stated definitions and postulates. One does not demand perfection in this regard; but in other works such as the *Elements* and the *Optics*, Euclid showed himself aware of the need for this accompanying appara-

[12] For details, see Menge 1916, xxxii–xxxiv.

[13] Thus, one has the same 'setting out' of the theorem to be proved, first in general terms and then with reference to parts named in a specific diagram, followed first by any necessary auxiliary constructions and then the proof proper, which ends with a statement of what has been proved, first in terms of the diagram and (sometimes) in general terms. (In the *Elements*, theorems, as opposed to problems, invariably end with the beginning of a general restatement of the theorem followed by the words 'and the rest.')

tus. In both these works the postulates are idealizations of experience that form the initial statements in a deductive system, and in both he states them without further comment.[14] In particular, the *Optics* opens with seven statements labelled 'definitions,' some of which are, however, clearly postulates.

In contrast, the *Phaenomena* opens with a discursive introduction in which we find a mixture of definitions[15] and statements that it is said must be assumed, including:

1. the stars are set into the surface of a single body and are carried around on circles which are everywhere equidistant from the eye;
2. the circles are all parallel, centering around a fixed pole star; and
3. the cosmos is spherical.

And the mixture of observations[16] and loose geometrical arguments introducing these statements is reminiscent of what one finds in an extant edition of the *Optics*, where some unknown editor later attached to the introductory 'definitions' an account of what Heiberg [1882, 138] has described as a lecture on the *Optics*. In it the lecturer attempts very much the same kind of thing as the author of the introduction to the *Phaenomena*, that is, to justify the basic principles by reference to everyday observations. These similarities lead one to suspect that the *Phaenomena* suffered the same treatment as the *Optics*.

Indeed, Neugebauer [1974, 756] has attacked the authenticity of the introduction to the *Phaenomena* on similar grounds, namely, that a treatise of a similar character, Theodosius' *De diebus et noctibus*, has an introduction and definitions which are only scholia, as is the case with the definition of uniform motion in *De sphaera*. Neugebauer also argues that much of the material in the introduction either repeats what is in the treatise or is irrelevant to it [see p. 51n33, below].

It should be said, however, that regarding the *Optics* and *De diebus* there can be no doubt that the introductions are spurious, given that the writer of the introduction to the *Optics* refers to the author in the third

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[14] The introduction to one of the extant recensions of the *Optics* is clearly not by Euclid. Heiberg assumed on insufficient grounds that it is by Theon, the industrious editor of the fourth century AD; but this is not widely accepted today.

[15] Defined are some common astronomical terms, such as 'ever-visible,' 'equator,' 'horizon,' 'meridian,' and 'tropics,' as well as such special notions as 'time of revolution of the cosmos' and 'the passage of an arc across the visible or invisible hemisphere.'

[16] 'Observation' must be understood not in a technical sense but vaguely, as in 'We've all seen this phenomenon, haven't we?'

person, and the writer of the introduction to *De diebus* refers to Theodosius; whereas there is no such clear evidence in the *Phaenomena*. Moreover, according to Aristotle's view of science, the introduction is more congruous with the rest of the *Phaenomena* than has heretofore been observed.[17] For, the terms θετέον and ὑποκείσθω, which the writer of the introduction uses to introduce the special assumptions of the work, remind one of the Aristotle's vocabulary in the *Analytica posteriora*; and, indeed, much in the *Phaenomena* is reminiscent of this work.

To begin with the vocabulary, in his *Analytica posteriora* Aristotle specifies two kinds of ultimate premisses in any given science. Some are axioms (ἀξιώματα)—called common notions (κοιναὶ ἔννοιαι) in Euclid's *Elements*—which contain principles used in demonstrations in all sciences. Others are θέσεις or principles peculiar to the science in question. These are either assertions that one of two mutually exclusive alternatives is the case (ὑποθέσεις) or definitions (ὁρισμοί). In the introduction to the *Phaenomena*, it is precisely the principles peculiar to this treatise that are introduced with the word θετέον.[18] They are not principles of mathematics, but of astronomy; and, indeed, Aristotle specifically mentions in *An. post.* i 13 that the science of the *phaenomena* is subsidiary to astronomy, this latter term being used indifferently to denote either mathematical astronomy or the practical sort sailors used. In this ranking of the sciences, it is the task of the lower science—in this case, the science of the *phaenomena*—to establish the facts and that of the higher science (mathematical astronomy) to establish the reasons for these facts.

Seen from this perspective, then, the available evidence suggests that Eudoxus' *Phaenomena* was concerned with stating certain facts of the case, and that Euclid's *Phaenomena* was concerned with establishing the reasons why the facts must necessarily be as they are. And the reasons are established in Euclid's work just as Aristotle prescribes, by demonstrations from the first principles (including the θέσεις) which show that, on the basis of such principles, the facts (in this case, the phenomena) are necessary.

How are the first principles to be established? At the end of *An. Post.* ii, Aristotle states that they are arrived at by repeated visual sensations which leave their marks in the memory. We then reflect on these memories and arrive, by a process of intuition (νοῦς), at the first principles. Now, this

[17] The authors are indebted to Henry Mendell in the following remarks about Aristotle's *An. post.*

[18] This is a form of the verb τίθημι, from which the Aristotelian θέσεις is also derived. The other word used in the introduction for assuming something, ὑποκείσθω, is a form of the verb ὑπόκειμαι which, according to Liddell and Scott, is used in Greek literature as the passive of ὑποτίθημι, a compounded form of τίθημι.

is what happens in the introduction to the *Phaenomena*, when the author justifies his θέσεις in terms of sensations whose memories he asks the reader to recall.

Thus, both in its vocabulary and evident purpose, the introduction to the *Phaenomena* fits well with the rest of the treatise. For, it complements the demonstrations contained in the propositions by arguing for the hypotheses on the basis of the phenomena (and by providing as well Aristotle's other sort of θέσεις, namely, the definitions) and then by showing how these work together to give the reasons for the phenomena.

Having said this, however, we should emphasize that the possibility of reading the treatise in Aristotelian terms does not imply that the writer was consciously following Aristotle's *dicta*. Indeed, R. McKirahan [1992] has recently argued that the mutual interaction between philosophic strictures and Greek mathematical practice was a complex one, and that the streams of influence by no means flowed in only one direction. Aristotle was working at a time of active mathematical practice, and his writings reflected as well as influenced that practice.

We should also emphasize that our observation that the introduction fits well with the treatise is not intended to settle the question of the introduction's authenticity. A suggestion about how a treatise might be read should not be construed as a historical argument for the authenticity of all its parts. It implies nothing stronger than the conclusion that, if the treatise were expanded by the inclusion of the introduction, it was done by someone who knew how to do it in a way that made sense from a philosophic (and perhaps pedagogic) viewpoint.

In the end it is not easy to decide on the authenticity of the introduction. However, we think that, on the balance of the evidence, the introduction did not belong to the treatise originally as written by Euclid. Authentic introductions to Hellenistic mathematical treatises from the time of Euclid to that of Apollonius have the form of letters.[19] Moreover, the *Phaenomena* became part of a corpus of material [see pp. 15–16, below] whose pedagogic aim rendered it a likely candidate for the addition of an introductory commentary, as Neugebauer pointed out happened in the case of some other members of this corpus. Moreover, the principal ancient commentator on the treatise, Pappus, makes no mention of the introduction, despite its length and the fact that it would have provided ample material for a writer

[19] An apparent exception to this statement is the introduction to the work on music theory, the *Sectio canonis*, which is not in the form of a letter, although the work has traditionally been included in the Euclidean canon. However, both editors (Menge and Jan) of the text of the *Sectio canonis* have doubted its attribution to Euclid in its present form.

like him to comment on. To go further towards resolving the doubts raised above one has to consider the history of the text and what it reveals about the origins of the present edition of the *Phaenomena*.

History of the text

Prior to H. Menge's publication of the *Phaenomena* in 1916, the best Greek text of the work was David Gregory's edition and translation of the complete works of Euclid published at Oxford in 1703. Of this edition, however, Heiberg [1882, 47] wrote:

> Now it strikes the eye immediately that the *Phaenomena*, in the form in which Gregory edited it, has been much marred by additions. For four of the eighteen theorems (6, 12, 14 and 15) other proofs are present, of which that for theorem 6 is indirect (which in this treatise is otherwise never the case) and the second proofs for 12 and 14 deviate in reality too little from the first for the two [proofs for each theorem] to be by the same author. Also there are, besides one denoted as such in the text (to Proposition 14), four more scholia in the text, which by the note 'moreover scholion' identify themselves clearly enough as trimmings (one to Proposition 12 and three to Proposition 14).

Gregory based his edition of the *Phaenomena* on a number of manuscripts; but, as with all manuscripts of the *Phaenomena* known at the time, they were members of a single family, whose oldest representative is the codex Vaticanus graecus 204.[20] Indeed, Heiberg wrote his *Litterargeschichtliche Studien* [1882] in part to call the attention of the scholarly world to the new information about the history of Euclid's works to be gained from a study of cod. Vind. gr. 31.13. In view of the merits of this codex and its relatives, all of which represented a single family, Menge denoted the version of the *Phaenomena* found in this family by the *siglum* **a**, and used **b** to denote the version found in the family best represented by cod. Vat. gr. 204.

There is no reason why Euclid should not have issued two editions (as they would now be called) of a work that is so clearly not the last word on its subject. (This is not to say that there are no detectable accretions, some of which are sufficiently extensive to merit inclusion in an appendix

[20] This family consists of some mss of texts of the Little Astronomy, described below; and its classification was worked out in Mogenet 1950. (Mogenet was concerned primarily with the mss of Autolycus, but his results should also be valid for the *Phaenomena*.)

of ten pages to Menge's edition.) However, none of the features of **b**, which we shall discuss below, suggests anything other than work by a later editor; and no writer on the subject has suggested that **b** represents Euclid's reworking of **a**.

Heiberg called the recension **a** 'far nearer to the original' than **b** because the scholia are separated from the text and the superfluous alternate proofs are not present. Heiberg also emphasized that the variations between the proofs in **a** and **b** ensure that neither **a** nor **b** is a source for the other.[21]

In our opinion, however, Heiberg overstates the conceptual differences in the proofs, for they amount only to the following. In Proposition 10, despite Heiberg's claim that the proof differs in conception, the only difference is that the author of **b**, misled perhaps by a diagram, has assumed unnecessarily that point A on the ecliptic must be the beginning of the sign Cancer. In Propositions 11 and 12 the differences, apart from details, lie in the fact that in **a** both have for the proof of their last parts only 'Similarly we shall prove that ... ' As for the proof of the Lemma—labelled 14 in certain manuscripts of **a** and appearing as a scholium in **b**—Heiberg again overstates the difference. While it is true that **a** proves the rising time of one arc is equal to the setting time of the other and **b** proves the setting time of one equal to the rising time of the other, there is no difference in the ideas behind the proofs. Finally, the proof in **a** of Proposition 16 (17 according to the numbering in **a**) lacks the necessary 'Similarly we shall prove ... ' found in **b**.

The presence of the full proofs of Propositions 11 and 12 in **b** is certainly consistent with the hypothesis that an editor has supplied details for a case that the author thought the reader could work out for himself (or that the teacher could work out for him). The misunderstanding in the proof of Proposition 10 might well be a sign of a careless editor; and, in general, none of the differences are such as to suggest that the two versions represent different editions of the work done by Euclid himself. That all manuscripts

[21] Heiberg's detailed comparison of other differences between **a** and **b** is as follows [1882, 50–51]:

> Through Proposition 8 they are essentially the same, and even in Proposition 9 the differences have the character of being only different readings. In 10 the proofs differ in conception and in 11 the proofs are quite different in detail. As well, in 11 the second half [in **b**] is replaced by a short sentence beginning 'Similarly.' The proofs of 12 differ in form. In 13 the proofs differ not only in approach but also in the lettering of the figures. The Lemma is numbered as a proposition, and has a proof different in conception. [In **a**] the proof of Proposition 14 (numbered 15) is much abbreviated, and the manuscripts break off shortly after the first part of the proof of Proposition 16.

of **a** break off midway through Proposition 16, argues that they all stem from a common archetype which may have lacked a page or been damaged at the end of a roll. The incomplete state of **a** would also explain why there are many fewer copies of it than of **b**, since a copyist would naturally pick a complete version if one were available.

Although Heiberg is surely right in insisting that cod. Vind. gr. 31.13 presents a version of the *Phaenomena* nearer the original than the text Gregory printed, this codex is by no means a faithful copy of the original edition, even if one puts aside the fact that it lacks Propositions 17 and 18 as well as half the proof of Proposition 16. Thus, in his commentary on the *Phaenomena*, Pappus [*Coll.* vi 104–129] has extensive notes on various points in the work; and in vi 104 he remarks that in Proposition 2 Euclid omitted any demonstration of the case when the zenith is on the tropics or between them. But all known manuscripts of the *Phaenomena*, including the Vienna codex and its relatives, discuss this case in Proposition 2; and, although it is possible that other copies of the text at Pappus' time had a discussion of the missing cases, it is simpler to assume that in Pappus' time there was still only one version of the treatise,[22] and that sometime later an editor completed Proposition 2 by including the cases mentioned by Pappus.[23]

By the time of Pappus, the *Phaenomena* had become part of a group of texts that provided a student versed in the requisite material from the *Elements* with the additional information necessary for the study of Ptolemy's *Almagest*. Indeed, the purpose of Pappus' *Coll.* vi was to comment on this material, referred to by an anonymous scholiast as the Little Astronomy. Since Pappus also comments on Autolycus' *De sphaera*, Euclid's *Phaenomena* and *Optics*, Aristarchus' *De magnitudinibus et distantiis solis et lunae* (*On Sizes and Distances of the Sun and Moon*), Theodosius' *Sphaerica* and the same author's *De diebus et noctibus*, we assume that the Little Astronomy included at least these six works. However, it seems to have come to include other works as well, since, for example, the three codices Vat.

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[22] On this hypothesis, the citation in Theodosius' *Sphaerica* of text that is unique to **b** would be a later interpolation in Theodosius' work. This would not be at all surprising, since Hellenistic mathematical writers, as opposed to commentators, seldom cite their predecessors within the confines of a mathematical treatise.

[23] If there were multiple versions prior to the time of Pappus, one would have to assume either that he did not know the ones which were the archetypes of our present versions and which bore the complete form of Proposition 2, or that emendations were made in different archetypes after Pappus' time so that they would all agree in the wording of Proposition 2. Neither of these hypotheses seems to us to be very likely.

gr. 191, 202, and 204 all contain, in addition to the above six works, three other treatises: Autolycus' *De ortibus et occasibus*, Hypsicles' *Anaphoricus*, and Theodosius' *De habitationibus (On Habitations)*—all dealing with very much the same sort of problems as the six treatises Pappus comments on. (And two more codices, Vat. gr. 203 and Parisinus gr. 2364, lack only the two treatises by Euclid which, in any case, we know from Pappus belonged to the collection.)

Hence, despite Neugebauer's reservations [1975, 768–769] about the existence, or importance, of such a collection, Mogenet [1950, 163–166], whose evidence we have summarized above, presents good reason for accepting as a reasonable interpretation of the evidence the thesis that a body of teaching material later known as the Little Astronomy was formed by at least the fourth century AD.

The history of the *Phaenomena* cannot be considered apart from the role of the Alexandrian commentators in mathematical education and, in particular, the history of the the Little Astronomy. For, an editor interested in using Euclid's *Phaenomena* for teaching would have found several problems with the work as Euclid wrote it. Its tacit use of non-trivial theorems of spherics and its use of such non-trivial astronomical notions as rising times without any explanation indicate that the treatise was originally intended for a learned readership and not as a text for beginners. Thus, its inclusion in the Little Astronomy necessitated not only a series of lemmas by Pappus but further changes as well in order to make it more suitable for instructional purposes. And among these changes was the addition of an introduction by the editor, one which would lead the reader gently into the abstract considerations of the text proper.

In seeking out a possible candidate for editor, Menge did not think it necessary to look much past Pappus' time, since Menge believed that Theon of Alexandria, who lived only about forty years after Pappus, edited not only Euclid's *Elements* and *Data*, but his *Optics* as well. Indeed, Menge, who was obviously aware of the problems Euclid's treatise raises, went so far as to state that it may have been for the purposes of inclusion in the Little Astronomy that Theon prepared the new edition, **b**, of the *Phaenomena*.[24] (In that case, if we accept Heiberg's contention that **a** is the earlier version, it was presumably Theon who incorporated into the text those scholia that are clearly separated from it in **a** and added the four alternate proofs.) He wrote that Theon's name 'comes to mind' as a possible compiler of **b** because Theon edited Euclid's *Elements*, *Data*, and *Optics*.

[24] Menge thus appears to echo Heiberg [1882, 51], who said that one is justified in ascribing the variants in **b** that are not simply copying errors to a later reworking that was probably undertaken for the Little Astronomy.

First of all, however, if Theon produced **b**, it is difficult to determine who was the editor of **a**. In our view, **a**, which we take to be the earlier version, had not been prepared by the time of Pappus. There is, after all, no evidence in Pappus' text that any of the alternate proofs found in **a** and incorporated into the text of **b** had yet been framed. If we are right, then, there is too short an interval between Pappus and Theon to produce edition **a**, with its alternate proofs and introduction, in time for Theon to rework it into **b**. And there are no likely candidates for the editor of **a** in the few decades between Pappus and Theon.

It seems to us more likely, therefore, that Pappus' commentary inspired the work which culminated in a later writer's producing version **a**, which represents Euclid's text after an early attempt at editing it; and that **b**—all of whose material can reasonably be seen as a reworking of that in **a**—represents a 'fleshed-out' version of **a**.

Furthermore, Menge's proposal that Theon edited the *Phaenomena*, quite apart from the question of which version, should be treated with the greatest caution. That Theon edited the *Optics* is, as we have noted, simply Heiberg's assumption. And the *Elements* and the *Data* are of a very different sort from the *Phaenomena*; thus, one can well imagine a mathematician's being interested in them without necessarily being interested in the *Phaenomena*.

On the other hand, Theon's interest in astronomy is evident from his commentaries on most of Ptolemy's *Almagest* and on his *Canones manuales* (*Handy Tables*). And even though a commentary would have been a very different thing from an edition of the *Phaenomena*, it does establish that Theon was deeply involved in the study and teaching of astronomical material. Moreover, Theon is the only individual we know of after Pappus who is demonstrably associated with any substantial work on the Little Astronomy. In fact, the anonymous author of a study of isoperimetric figures [see Hultsch 1887–1888, 1142] cites him as the author of a commentary on the Little Astronomy. Finally, the dates are right. As we have already noted, there are grounds for maintaining that the *Phaenomena* was first edited after Pappus, who wrote half a century earlier than Theon. Thus, in our view, Theon may have been the source of version **a** of the *Phaenomena*.

In conclusion, we propose that Pappus had a version of the *Phaenomena* in substantially the form in which Euclid wrote it, and that he then composed the notes to aid in its teaching that are found in his *Coll.* vi. At some later time, an editor wrote an introduction to the work, changed Proposition 2 to take into account Pappus' notes, and attached some alternate proofs which had appeared after Pappus' time and perhaps some further additions to produce what was essentially **a**. Still later, the text was further

reworked to incorporate the alternate proofs and some scholia, resulting in our version **b**. Subsequently, the end of the earlier version was lost; and so **b** became the more frequently copied edition, even though enough of the earlier remained for it to be worth copying and (hence) preserving as our version **a**.

2

Euclid's Presuppositions

Mathematical presuppositions

Euclid's mathematical presuppositions concern the geometry of circular arcs on the surface of a sphere. They begin with the ideas of circles, semicircles, and spheres. These basic notions are specified in *Elem.* i defs. 15, 18, and xi def. 14. In the *Phaenomena*, as in his *Elements*, but in contrast to modern usage, Euclid clearly intended circles and semicircles to be taken as two-dimensional figures, and spheres as three-dimensional solids.[1]

As for assumptions that pertain to spherics proper, in the course of proving his theorems Euclid tacitly assumes several results in the geometry of the (fixed) sphere, results either equivalent to, or consequences of, theorems found in Autolycus' *De sphaera* or Theodosius' *Sphaerica*.[2] We shall list here the major theorems Euclid assumes, along with references to ancient works in which they may now be found. In citing a work, however, we by no means wish to imply that it was Euclid's source, nor even that he had a single source. After all, although Euclid's date is uncertain, he certainly lived before Theodosius (*fl.* ca. 150 BC); and so he could not have cited the *Sphaerica*. Moreover, even if Euclid drew on Autolycus' *De sphaera*, it certainly does not contain enough of the results he needs to have been his only source.

[1] For example, in Propositions 4 and 5 Euclid refers to the 'circumference' of circles (meaning their boundaries), which would make no sense if circles were simply curves.

[2] The basic reference is Hultsch 1886: cf. Berggren 1991.

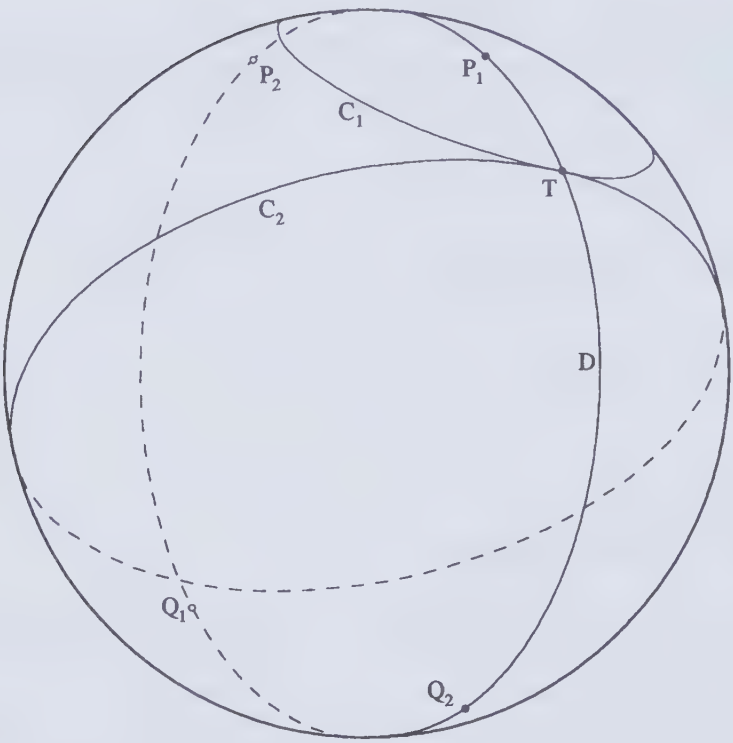

Figure 2. Tangency Theorem
(Dashed arcs and hollow dots are on the rear hemisphere.)

One elementary result that Euclid supposes the reader to know is that when a plane cuts a sphere, and is not just tangent to it, the intersection is a circle whose circumference is on the surface of the sphere [cf. Theodosius, *Sphaer.* i 1]. Another is that, when the circle contains the centre of the sphere, its centre coincides with that of the sphere and, in this case, according to *Sphaer.* i 6, the circle is the greatest circle in the sphere. (We shall refer to these greatest circles simply as great circles.) Euclid also assumes that, since any two great circles in a sphere intersect in a line which necessarily passes through the centre of each, any two such great circles bisect each other [cf. *Sphaer.* i 11].

However, when the plane intersecting the sphere does not contain the centre of the sphere, it forms not a great circle but one parallel to a unique great circle. (In the case of Euclid's *Phaenomena*, such circles are all parallel to the equator.) It is a consequence of Theodosius, *Sphaer.* i 1 that the centres of parallel circles in a sphere all lie on the diameter of the sphere

perpendicular to any one of them. And *Sphaer.* i 8 shows that such a diameter intersects the surface of the sphere in a pair of diametrically opposite points, called poles of the parallel circles; the line joining the two poles is called an axis of the sphere. In Euclid's *Phaenomena*, two pairs of poles come into consideration. The first pair are those of the celestial equator, that is, the north and south celestial poles. The north pole is simply called the visible pole of the sphere; the south pole is not mentioned separately. The line joining them is what Euclid means by 'axis.' The second pair consists of those of the horizon, now called by the terms of Arabic origin, 'zenith' and 'nadir,' but having no special name in Euclid's work.

The foregoing results and terminology pervade the *Phaenomena*, but there are other propositions Euclid presupposes in the proofs of specific theorems of his work. Among them is a consequence of Theodosius, *Sphaer.* i 13 and 15 which we shall call the Perpendicularity Theorem:

> A great circle is perpendicular to a given circle of a sphere if and only if it passes through the poles of that circle.

Another, hereafter the Tangency Theorem [see Figure 2], is equivalent to *Sphaer.* ii 5:

> If two circles C_1 and C_2 on a sphere are tangent,[3] then the great circle D through the point of tangency T and the poles of one, P_1 and Q_1, also passes through the poles of the other, P_2 and Q_2.

Euclid also uses throughout the treatise a fundamental theorem that is equivalent to *Sphaer.* ii 17 and 18 which concern two circles parallel to a great circle C and on either side of it. These two circles are equal if and only if they cut off from some other great circle equal arcs on either side of C. We shall refer to this in the sequel as the Theorem of Parallels.

Another elementary theorem that serves, for example, in the proof that arc ED is equal to arc DM in the second part of the proof of Proposition 7, is that the two arcs of a great circle cut off between two parallel circles that intersect it are equal. (In Proposition 7 one of the two parallel circles is tangent to the great circle.)

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[3] Although the authors of both the *Phaenomena* and *De sphaera* use the notion of tangent circles on a sphere, neither of them defines it. Still, to judge from the arguments in these treatises, they simply regarded them as circles with only one common point. At the beginning of *Sphaer.* ii, Theodosius, however, defines two circles as tangent at a point if the line through that point and in the plane of each circle is a tangent to each circle. See Berggren 1991, 241–243, for discussion of the development of the idea of tangent circles on the sphere from the time of Autolycus to that of Theodosius.

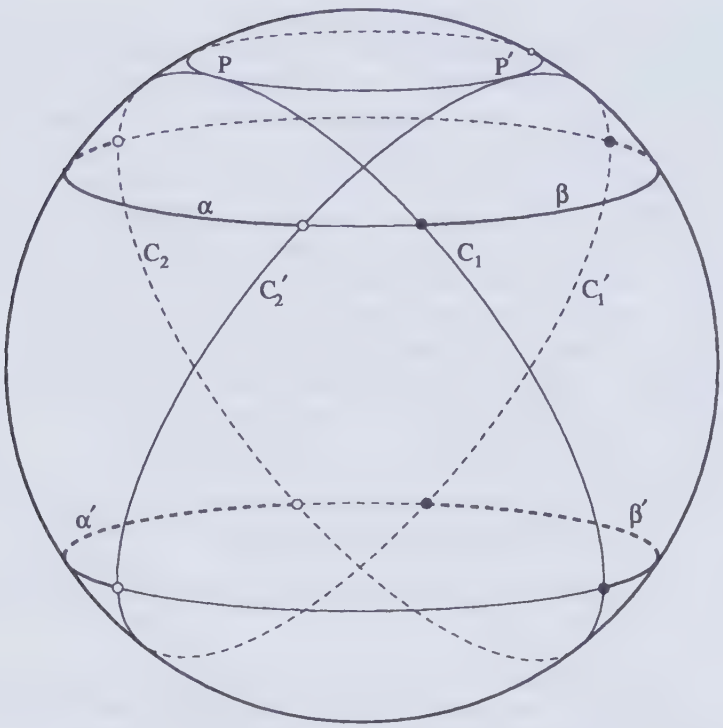

Figure 3. The Asymptotic Semicircles Theorem
(Correspondingly shaded dots mark the ends of similar arcs on parallel circles and of equal arcs on great circles.)

Yet another theorem much used by Euclid is equivalent to Theodosius, *Sphaer.* ii 13, which concerns what the Greeks called asymptotic semicircles of two great circles of the sphere. One imagines two great circles C and C' tangent to a given small circle at points P and P' (respectively). (Intuitively, one may regard C' as an image of C under a rotation of the sphere around the poles of the small circle by something less than a half turn.) Let us denote by C_1 and C_2 (C_1' and C_2') the two semicircles of C (C') that begin at P (P'), where C_1' and C_2' are rotated images of C_1 and C_2, respectively. Then, C_i intersects C_i' only at the centre of the sphere, and C_i and C_i' are what the Greek authors meant by 'asymptotic semicircles' of C and C'.[4] *Sphaer.* ii 13 then states what we shall call the Asymptotic Semicircles Theorem [see Figure 3]:

[4] Although the Greek word we translate as 'asymptotic' means 'not intersecting', Euclid's asymptotic semicircles clearly do intersect each other at the centre of the sphere, rather like two crossed knife blades intersecting at a point. Yet Euclid's

If C and C' cut circles parallel to the one they are tangent to, then the arcs of these parallel circles between the asymptotic semicircles are similar and corresponding arcs of the great circles between any two of the parallel circles are equal.

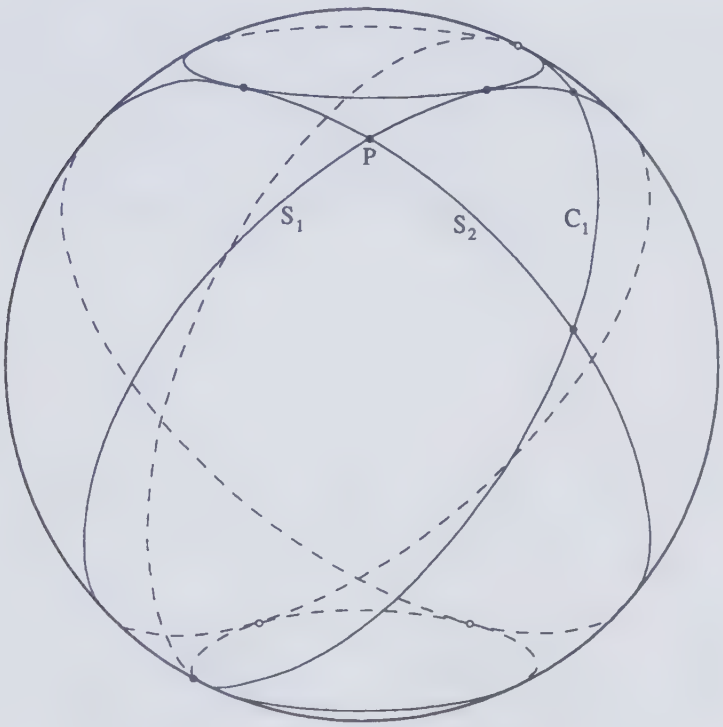

Figure 4. The Tangent Circle Theorem

Also important in this context is a theorem equivalent to *Sphaer.* ii 15 which affirms that, given a point P between two equal parallel circles on a sphere, it is possible to draw two great circles, S_1 and S_2, tangent to the parallel circles and passing through the given point. Euclid evidently had in mind a consequence of this, the Tangent Circle Theorem [see Figure 4]:

If C is any circle tangent to the two equal parallels that does not pass through a given point P, and if C_1 is one of its two semicircles

use of 'asymptotic' is excusable because the semicircles do not intersect where it matters, that is, on their curved circumferences. In effect, the Greek term as Euclid deploys it is a technical term for which there is no equivalent term in English. So we have used the cognate English technical term rather than attempt a literal translation.

going from one parallel circle to the other, then one of the two constructible semicircles, S_1 and S_2, through the given point and tangent to the parallel circles is asymptotic to C_1. (In Figure 4 it is S_1.)

In *Sphaer.* ii 19 Theodosius introduces another great circle H which cuts some but not all of a family of parallel circles, for instance, $AA'A''$, $BB'B''$, and $CC'C''$. Thus, H does not go through their poles; so each of the two hemispheres into which H divides the sphere contains one of the poles of the parallel circles. One of the hemispheres is called visible and the other, invisible, so that H serves as the horizon even though Theodosius never calls it such. The conclusion of ii 19, hereafter the Visibility Theorem [see Figure 5], is:

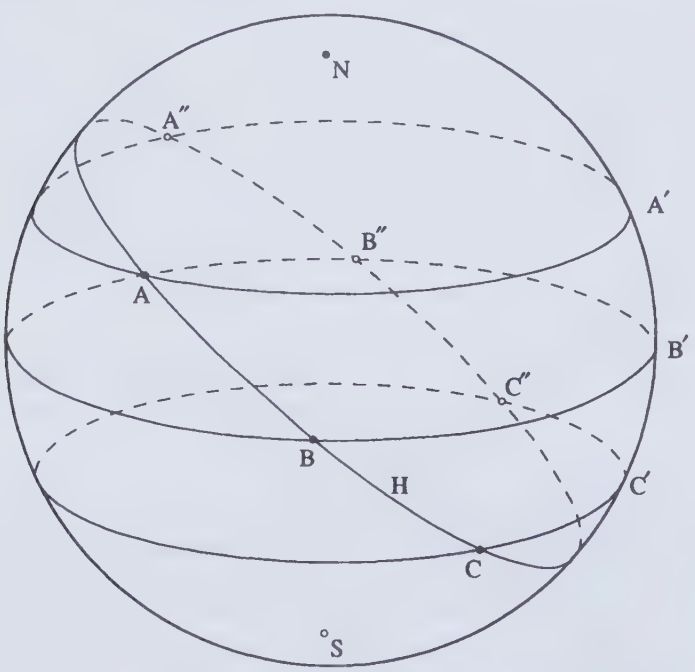

Figure 5. The Visibility Theorem

The nearer a parallel circle cut by H is to the visible pole N, the larger proportion its visible segment is of the whole parallel circle; and the nearer it is to the invisible pole S, the larger proportion its invisible segment is. Also, given two equal parallel circles, the segment of one of them in the visible hemisphere is equal to the segment of the other in the invisible.

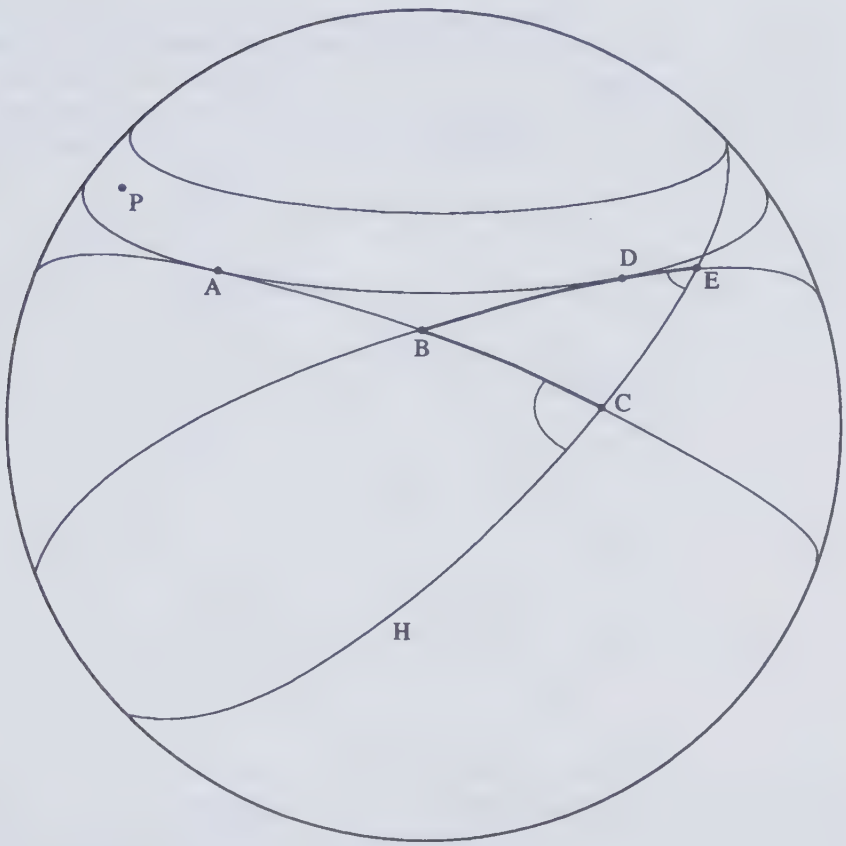

Figure 6. The Inclined Arcs Theorem

Euclid also presupposes another theorem, which we shall call the Inclined Arcs Theorem [see Figure 6]:

> Suppose a great circle H is tangent to one of two parallel circles and cuts the other, which is parallel to the first and lies between it and the center of the sphere. Suppose also that P, the pole of H, lies between the two parallel circles. Now, let there be given two other great circles, ABC and BDE, that are tangent to the larger of the two parallels at A and D and intersect at a point B—the points A, D, and B being on the same side of H as P—and suppose that C and E lie on H. Finally, suppose that A lies nearer than D to the midpoint of the larger of the two arcs into which H divides the larger of the parallel circles. Then, arc ABC makes a larger angle with H than does BDE and arc BDE is longer than arc BC.

The hypotheses of the Inclined Arcs Theorem, with the conclusion about the angles, are the hypotheses and conclusion of *Sphaerica* ii 22; and one can see, at least intuitively, that the statement about the arc lengths follows from the statement about the angles, for since arc BC makes a larger angle with the horizon than does BDE, the latter arc should be the shorter.

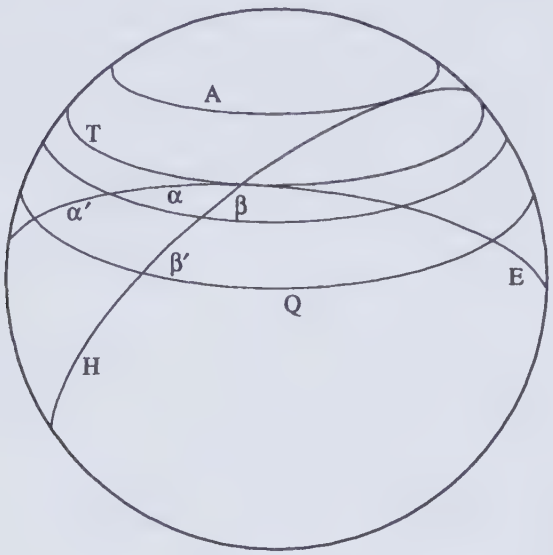

Figure 7. The Horizon Arcs Theorem

(This theorem applies to the projection of equal arcs of the ecliptic onto the horizon by means of circles parallel to the equator. Such projections determine the arcs of the horizon over which the various arcs of the ecliptic will rise, that is, their ortive or rising amplitudes.)

Finally, Euclid uses extensively an equivalent of Theodosius, *Sphaer.* iii 7 and 8, which deal with a great circle H (for Euclid, the horizon) tangent to a small circle A (for Euclid, the largest of the ever-visible circles). In addition, another great circle E (Euclid's ecliptic) is tangent to a small circle T (for Euclid, the summer tropic) parallel to A but larger than it; and E is so situated that H passes through the point of tangency of E and T. Finally, two equal consecutive arcs are cut off on the great circle E between its point of tangency with T and its intersection with the great circle Q (Euclid's equator) parallel to T. The conclusion of iii 7, the Horizon Arcs Theorem [see Figure 7], is:

> The parallel circles through the end points of these arcs will cut off on H (the horizon) unequal arcs, the larger being the one nearer Q.

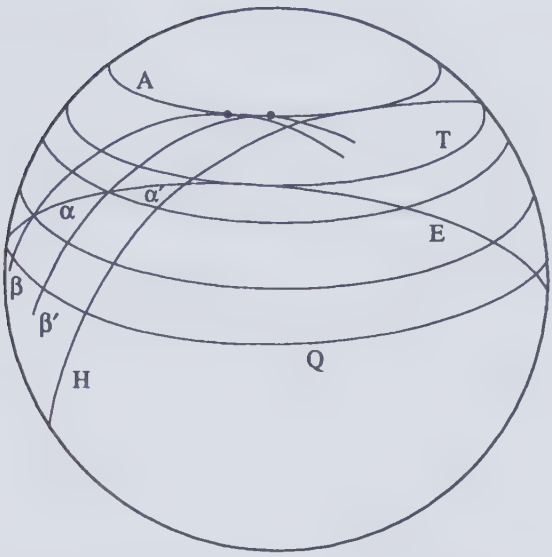

Figure 8. The Equator Arcs Theorem

(This theorem concerns the projection of arcs of the ecliptic onto the equator by means of circles representing successive positions of a horizon; thus, it concerns rising times.)

Theodosius, *Sphaer.* iii 8 adds the additional requirement that one take great circles tangent to A which pass through the endpoints of the two equal arcs so that their semicircles which go from the points of tangency with A through the end points of the two arcs are asymptotic. The conclusion of iii 8, the Equator Arcs Theorem [see Figure 8], then, is:

> These semicircles will bound unequal arcs on the equator Q, and that arc nearer to H will be the longer.

The complexity of the assumptions made about the geometry of the sphere is clear evidence that Euclid was not writing this treatise for beginners. We shall draw attention to various other presuppositions, as well as specific instances of those already mentioned, in the commentary to our translation of the *Phaenomena*.

Astronomical presuppositions

Euclid takes for granted that the reader is familiar with what is now called the two-sphere model of the cosmos,[5] according to which the cosmos is contained in an immense sphere at whose centre is the Earth and which is bounded by the sphere of the fixed stars. The Earth is conceived of as a sphere so small, in relation to the cosmos, that, as Ptolemy says [*Alm.* i 6], it has the ratio of a point to the size of the sphere of the fixed stars.[6]

An immediate consequence of this model is that the answer to the question of which stars are above the earth,[7] is relative to the latitude of the observer, since observers at different latitudes see different stars above the earth. The answer also depends on the time: for, as noted in the introduction to the *Phaenomena*, stars are always rising and setting. However, as the introduction also maintains,

> A certain star between the Ursae is seen not to change places but to revolve in the place where it is.

This particular star [see pp. 48–49, below] and the centre of the Earth define a diameter of the sphere called its axis, whose end points on the sphere of the fixed stars are the north and south celestial poles. The north

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[5] See Kuhn 1957. Strictly speaking, only the assumption of a spherical cosmos is made explicit in the *Phaenomena*; however, the assumption that the Earth is also spherical is implicit in such phrases as 'when the zenith is between the summer tropic and the visible pole' since including this condition makes no sense if every locality has the same zenith.

[6] About 400 years before Ptolemy, Archimedes, in his *Arenarius* (*Sand Reckoner*), made the same point about the Earth in relation to the sphere containing the Sun's orbit (which was what he said most astronomers called 'cosmos'). Numerical estimates of the size of the cosmos varied widely. Eratosthenes of Cyrene was, reportedly, the first to give a mathematical argument for the size of Earth, namely, 250,000 stades in circumference (rounded to 252,000 in order to be divisible by the 60 units into which, according to Strabo, Eratosthenes divided the circle.) In the *Arenarius*, Archimedes deliberately takes estimates which will force the radius of the sphere of the fixed stars to be very large, and arrives at the upper bound of 100,000,000 times the Earth's radius. However, according to a third-century Bishop of Rome, Hippolytus, Archimedes calculated the distance from the surface of Earth to the fixed stars as 248,264,780 stades (that is, 5,000 to 6,000 times the radius of the Earth) [cf. Neugebauer 1975, 648].

[7] Although we usually translate Euclid's γῆ as 'the Earth', we render ὑπὲρ γῆς by 'above the earth', in order to clarify the distinction between remarks about the celestial body and remarks about the local horizon, that is, the circle on the surface of the Earth that is in principle visible to an observer at any point on it.

pole is the one referred to as being between the Ursae, and the south pole is the one not visible from the latitudes inhabited by the classical authors.

If observers in localities where the north pole is visible stand facing the north pole, their backs will be to the south and they will see the stars rise in the eastern half of the horizon to the right and set in the western half on the left. The line through each observer and perpendicular to the north-south line defines by its intersections with the eastern and western horizons the local east and west points. The daily rotation of the stars and other heavenly bodies as seen from a given locality is from east to west.

In the course of this rotation the stars and other celestial bodies trace out parallel circles, all centred on the axis through the poles, and the largest of these parallel circles is called the equator. It is argued in the introduction that the equator is a great circle, so that it divides the celestial sphere into two hemispheres. Euclid describes stars in the hemisphere containing the north pole as 'toward the north,' and those in the hemisphere containing the south pole as 'toward the south.' The equator passes through the local east and west points; and like the other parallel circles its position in the sky does not vary during the daily rotation.

Regarding the parallel circles traced by the daily rotation of the stars, Euclid presupposes that:

> If a sphere turns uniformly around its axis, all of the points on the surface of the sphere traverse similar arcs of the parallel circles on which they are carried in equal times. [cf. Autolycus, *De sphaera* Theorem 2]

Another important great circle on the celestial sphere is the ecliptic.[8] The Sun appears to move around this circle during the course of a year in a direction opposite to its daily motion—that is, counterclockwise when viewed from the north—at an average rate of a little less than one degree each day. It traverses the ecliptic's twelve equal arcs or zodiacal signs[9] (as in Figure 1) in the order: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius, and Pisces. The leading point or beginning of any arc of the ecliptic is the initial point of the arc when it is traversed in the direction of the Sun's annual motion; it is also the first point of the arc to rise over the horizon in the diurnal motion of the

[8] This circle, though variously designated in the Greek text, is never referred to as ἐκλειπτικός. See p. 45n16 below, and the entries in the English Glossary.

[9] Originally, the ζῴδια (statuettes, figurines) were zodiacal constellations: later, they were also the twelve 30°-arcs of the ecliptic loosely associated with these constellations.

heavens. (The arc's trailing point is correspondingly defined.) The ecliptic is oblique or tilted relative to the equator; so that the six signs—Aries, Taurus, Gemini, Cancer, Leo, and Virgo—lie to the north of the equator and the other six lie to the south. In fact Aries and Libra are so situated that their leading points are on the equator. When the Sun in its passage to the northern signs is on the leading point of Aries, it is the day of the vernal equinox; and when the Sun, in passing south of the equator, is on the leading point of Libra, it is the the day of the autumnal equinox.

When Euclid refers to the semicircle following Cancer (or Capricorn), he means the semicircle defined by the signs Cancer, Leo,... Sagittarius (or Capricorn, Aquarius,... Gemini). Likewise, any arc in one of these semicircles is said to follow another arc in the semicircle if it comes after it in the appropriate sequence.[10]

Since the ecliptic is oblique to the equator, with which it makes an angle of about 24°,[11] there is obviously a most northerly point and a most southerly point on the ecliptic. In Euclid's day, the most northerly point was located at the beginning of the zodiacal sign of Cancer (where the Sun was at the beginning of summer); and the most southerly, at the beginning of the sign of Capricorn (where the Sun was at the beginning of winter). Since the ecliptic rotates daily along with the rest of the cosmos (except for the Earth), these two extremal points trace out circles that are parallel to the equator and are known, respectively, as the summer and winter tropics.[12]

Concerning the ecliptic and tropics, Euclid assumes in Proposition 7 that:

> In a sphere, if a great circle [the horizon] is inclined to the axis [of the sphere] and bounds the visible and invisible [portions] of the sphere, and another inclined great circle [the ecliptic] touches larger circles [tropics] than does the bounding circle, then it [the ecliptic] rises and sets at all points of the bounding circle between the parallel circles it touches [*scil.* the tropics]. [cf. Autolycus, *De sphaera* Theorem 11]

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[10] On the same idea, but different terminology, in Ptolemy's *Almagest*, see Toomer 1984, 20.

[11] According to Ptolemy, this angle was assigned the value 23° 50′ by Hipparchus (ca. 150 BC). Ptolemy, who lived some three centuries later, proposed the value, 23° 51′ 20″.

[12] These circles are defined differently in the introduction to the *Phaenomena*. The definition given here is essentially that found in Geminus' *Introductio astronomiae* (*Introduction to Astronomy*), with the difference that, for Geminus, it is not the leading points of Cancer and Capricorn but the Sun when it is at these points that traces out the tropic circles in the course of one revolution of the cosmos.

At any place on Earth there is a unique plane, the horizon, passing through the eye of the observer at that locality and separating the visible part of the cosmos from the invisible.[13] And this plane, according to an argument in the introduction, is in fact a great circle within the sphere. Indeed, as Euclid reminds his readers in Proposition 1, the Earth is as a point when considered in relation to the rest of the cosmos; consequently, one may suppose that the observer is in fact at the centre of the cosmos and, hence, that the horizon divides the celestial sphere into two equal parts. Thus, for a particular locality, the horizon is a fixed great circle whose inclination to the equator (which is the complement of the observer's latitude, a concept not used at Euclid's time) determines what particular phenomena an observer at that locality may see on the celestial sphere.

In the course of arguing that the Earth is at the centre of the cosmos [cf. Proposition 1], Euclid again tacitly assumes a theorem that is proven by Autolycus [*De sphaera* Theorem 12: but cf. pp. 54–55, below]:

> If in a sphere a fixed circle always bisects a moving circle, and if neither of the circles is perpendicular to the axis nor contains the poles, then each of them is a great circle.

Of course, the positions of the poles of the horizon circle, those two points on the celestial sphere directly above and below the observer, vary with the position of the observer on the Earth. Indeed, only for an observer at the north or south poles do they coincide with the poles of the sphere. Propositions 2 and 14 consider the position of these poles in relation to various circles parallel to the celestial equator in order to define a range of permissible latitudes for localities on the Earth's surface.

Concerning the horizon, Euclid takes for granted that

> If a great circle in a sphere that is inclined to the axis bounds the visible and invisible [portions] of the sphere, then a circle through the poles of the sphere is twice perpendicular to the bounding circle in one rotation of the sphere. [cf. Autolycus, *De sphaera* Theorem 10]

Geographical presuppositions

This treatise furnishes a revealing picture of early Hellenistic geography: for, although an observer's longitude is irrelevant to phenomena discussed

[13] So it is defined in the introduction to the *Phaenomena*; but it may also be described as the plane tangent to the spherical Earth at that observer's locality.

in Euclid's treatise, latitude is often of the essence; and how the positions of an observer are specified shows how far Greeks of Euclid's time were from our (and, indeed, Ptolemy's) equatorially based system. For Euclid's contemporary readers, places were specified by determining their zeniths in relation to the tropics or arctic circle, or by comparing the sizes of the tropic and arctic circles.

In the *Phaenomena*, Euclid is writing for persons living where the celestial north pole is visible, that is, for those living in the northern hemisphere. Admittedly, the explicit occurrence of this assumption in the statement of Proposition 9 in recension **a** may be, as Menge [1916, 44] suggests, an interpolation. Moreover, since we doubt the authenticity of the introduction, the implicit assumption there that the observer is in the northern hemisphere is of little consequence for our argument. However, it is explicit in Proposition 2[14] (resp. 14), which puts the zenith in a relation to certain parallel circles that is equivalent to the condition that $\epsilon < \phi$ (resp. $\epsilon < \phi < 45°$), where ϵ denotes the inclination of the ecliptic to the equator and ϕ, the local latitude.

That these conditions are equivalent to those in Propositions 2 and 14 is easy to see. The elevation of the most southerly point of the equator over the horizon is $90° - \phi$; so the elevation of the most southerly point of the summer tropic over the horizon is $(90° - \phi) + \epsilon$. The condition in Proposition 2 requires that the zenith lie north of the summer tropic. This means that $(90° - \phi) + \epsilon < 90°$, which is equivalent to $\epsilon < \phi$. Proposition 14 adds the condition that the zenith be south of the arctic circle, whose circumference is ϕ south of the north celestial pole. Thus, $2\phi < 90°$, that is, $\phi < 45°$.

As the last example suggests, it also happens that the observer is assumed not to be too far north. Thus, in Propositions 6 and 9, Euclid posits without comment an observer for whom $\phi < 90° - \epsilon$; and again, in Proposition 7, he makes an explicit assumption about the relative sizes of certain parallel circles that is equivalent to the same condition.

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[14] However, an interpolation in this theorem allows the zenith to be as far south as a point on the winter tropic.

3

Notes on the Translation

The Greek text

We have made our translation from the Greek text established by H. Menge [1916] from six codices, five of one recension (**a**) and one of the other (**b**). Menge examined many other codices for **b**, which differs from **a** only after the eighth of the eighteen propositions; and he concluded that all the codices for **b** derive from the same source, best represented by the codex Vat. gr. 204, 'by far the oldest and most faithful' [1916, xix] for that recension. Moreover, he found that a single document, codex Vind. gr. 31.13, 'easily first both in age and authority' [1916, xvi] best represents **a**. In spite of its incompleteness—this manuscript comprises only slightly more than the first fifteen theorems—Menge prefers **a**, and prints **b** in smaller type when **a** is available.[1]

Other translations

The first printed Latin translation of the *Phaenomena* appeared in a translation of Euclid's works by Bartholomaeus Zambertus in 1505 in Venice. In 1558 Euclid's treatise was published in another Latin translation by Maurolycus, who evidently made use of an Arabic version, along with the other ancient works on spherical astronomy or spherics with which it had long been associated. Next, Joseph Auria's Latin translation was published in Rome in 1591; this translation followed its predecessors and

[1] On the recensions and their authenticity, see pp. 13–15, above.

even incorporated some of Maurolycus' notes. It was, however, only with the David Gregory's Latin translation [1703] that the Greek text was first printed, a text inferior to that prepared by Menge. Still, Gregory's text was the basis for the only translation of Euclid's *Phaenomena* into a modern language that we are aware of, namely, a nineteenth-century German translation (without the introduction) by A. Nokk [1850]. Up until now, only Thomas Heath's translation [1932] of almost the whole introduction and the enunciation of Proposition 1 has been available in English.

General remarks

Our translation is based on recension **a** up to the end of Proposition 8 and, again, in the lemma between Propositions 13 and 14; on both recensions **a** and **b** for Propositions 9 to 16; and on recension **b** for Propositions 17–18.

In order to maintain a contemporary style in our English version of the *Phaenomena*, we have often used several sentences or paragraphs to translate one sentence or one paragraph in the edited Greek text; for example, we render the first sentence of the introduction in Menge's text by three rather lengthy sentences in our translation. Treating Greek sentences and paragraphs in this way requires some surgery on particles and conjunctions, however. The most drastic occurs when a single Greek sentence takes up a substantial portion of a page, and contains many subordinate clauses governed by 'since' and followed by a conclusion. In some of these cases, we promote the subordinate clauses to main clauses and cast the conclusion in another main clause beginning with 'it follows that' [cf. Proposition 7].

In the translation, square brackets [ ] indicate additions we have made for the sake of clarity. Pointed brackets < > enclose text that appears in Menge's edition in square brackets as interpolations. These interpolations have been translated when they seem to help, and omitted when they seem pointless, as when a sentence is repeated [Menge 1916, 60]. Parentheses ( ) are inserted merely to improve clarity, as is the use of italics. All punctuation follows contemporary English usage, though much of it still reflects the periods, commas, and colons of the edited Greek text.

There are elliptical expressions in any Greek text. Many here have been completed in our translation within square brackets. One type of ellipsis that is extremely common in the *Phaenomena* occurs in lists of arcs or points. In this case, we have added '[and]' before the final entry, since other ways of dealing with such lists seem odd to us or sometimes misleading.

The reader is advised that when Euclid names an arc, the order in which the ends are specified is of no significance. (If an intermediate point is

mentioned, then it is probably placed in the middle of the list.) A pair of lists of points, arcs, and so on, where one might expect the word 'respectively' to follow the latter list, will not have it because it is never in the Greek and because it may not be true. For example, in the proof of Part 1 of Proposition 12 in **a**, though 'the semicircles through S [and] T in the directions of C [and] F' might lead one to expect C to be on the semicircle through S, it is in fact on the semicircle through T.

There is a feature of the proofs offered in the *Phaenomena* that may have been as puzzling to the original readers as it is today, namely, the frequent promise of proofs yet to come. The oddity is that such promises are never fulfilled. One possible explanation of this is that the missing proofs will be supplied on demand; so that there is, in effect, an ellipsis of 'if required.' Another explanation is that the promise amounts to a statement that proofs are possible. We have decided to translate these promises as literally as we can, in order to preserve in English this puzzle in the original.

For simplicity, we have introduced two technical terms into the translation that capture the meaning of the Greek but do not represent the Greek words actually used. Thus, we call the point on the celestial sphere directly over the head of the observer the observer's zenith, using the modern term derived from Arabic. (It is also the visible pole of the observer's horizon, and that is how it is always referred to in the Greek, but without the addition of 'visible.') In the case of the circle on the celestial sphere representing the path of the centre of the solar disc, this circle is designated in a variety of ways in the introduction to the treatise but consistently thereafter as the ζῳδιακὸς κύκλος, a phrase that might be translated as 'zodiac(al) circle' if that expression had any clear sense in English. We have preserved the variety of the expressions in the introduction by translating the Greek literally, but have decided to render ζῳδιακὸς κύκλος by 'ecliptic.' Our decision should not be taken to entail that Euclid associated this circle with eclipses—we have no idea whether he did.

Technical remarks

On the whole, we have attempted to be consistent in translating Greek expressions. There are occasions, however, where we have abandoned consistency in favour of systematic inconsistency, especially in terms involving the comparison of size and extent. Thus, for intervals of time we have used 'greater,' 'greatest,' 'lesser,' and 'least' to translate the comparative and superlative forms of the adjectives πολύς and ἐλαχύς [cf., e.g., Menge 1916, 48a.1 and 6, 44a.12]. Moreover, whereas we have rendered the comparative and superlative forms of μέγας and ἐλαχύς by 'larger,' 'largest,' 'smaller,' and

'smallest' in the case of circles and segments of circles [cf., e.g., Menge 1916, 6.4, 2.21–22, 4.5]; we have preferred to render the same words by 'longer,' 'longest,' 'shorter,' and 'shortest,' in the case of circular arcs [cf., e.g., Menge 1916, 48a.4, 42.7, 66b.8]. For, it seemed to us clearer and more digestible to use some of the variety available in English systematically to vary the diet, in particular because of the special meaning of 'great' (μέγιστος) when applied to circles of the sphere.

Euclid often has occasion to say that two times are equal, and he usually does this by affirming that one rotation takes place in the time in which another rotation occurs. It is ambiguous in the Greek—as in the English—whether only equality of duration or simultaneity is meant as well. In cases when it appears that the time-periods are simultaneous, we have translated ἐν ᾧ χρόνῳ (in the time in which: cf., e.g., Menge 1916, 16.9) by 'while.' But, when ἐν ᾧ χρόνῳ seems to us to mean 'at the time when' [cf., e.g., Menge 1916, 16.19], we have rendered it simply by 'when.'

One often finds in Greek 'the X' where we must say in English either just 'X' or 'point X.' Now, in the proofs of the *Phaenomena*, a single letter always signifies a point and two letters usually signify an arc. The important exceptions occur in the proofs of Proposition 12, where line segments are meant but not indicated in the Greek except by the gender of the articles. Accordingly, in our translation, we often depart from usage in the *Phaenomena* by not specifying points and arcs as such. Moreover, when we do add the appropriate substantive for the sake of clarity, we enclose it within square brackets.

The transliteration scheme used in labelling points is intended to be mnemonic for those who know the Greek alphabet. The twenty-four capital letters of the classical Greek alphabet are represented respectively by A B G D E Z H Q (= Θ) I K L M N X O P R S T U F C J (= Ψ) W. Koppa (ϙ) in the proof of Proposition 14 is represented by V. The few lowercase letters correspond to the uppercase ones. The one exception is that we have used s for ϛ (digamma), which appears in the diagram on Menge 1916, 65 and throughout the proof of Proposition 12 in recension **b**, since σ does not serve as a label in the *Phaenomena*.

We note here a few misprints in the edited Greek text. In Menge 1916, 36.13, read Z for Σ; and in 42.4, 22, read ΘΛ for ΘΔ and ΛZ for ΔZ. (Menge's Latin translation is correct in all instances.) At Menge 1916, 81, the point H in the figure is marked as being off the curve EZ, when it should be on it. In Menge 1916, 82b.3–4, arc OP has a masculine article, when it should be feminine. In light of our reading of the text, we have also moved P and R in our Figure 17 from their positions in the diagram accompanying Proposition 7 [Menge 1916, 37].

The figures

As Neugebauer [1975, 751] points out, the diagrams are vital to understanding the theorems. But, as Menge [1916, xvi] has noted, the figures in the manuscripts are not good and are much corrected. Indeed, Menge reports that the diagrams were badly copied, if not omitted altogether (as in the least complete manuscript of recension **a**). Moreover, it would seem that successive copyists have tampered with the labels in the diagrams in order to suit their interpretations of the text.

We have taken to heart Neugebauer's remark [1975, 753] that 'in many cases it is misleading to explain an ancient theorem with the help of a perspective drawing.' Some experimentation has convinced us, however, that Neugebauer's preferred solution, stereographic projection, would in various instances be neither appropriate nor helpful to the reader. Thus, we have elected to follow the diagrams in Menge's text.

These diagrams vary accordingly as the arctic circle does or does not play a role in the argument. In the latter case, the diagrams present a crude perspective representation of the celestial sphere as viewed from the outside [cf., e.g., Propositions 6–11 and 13]. In such representations, the circles on the sphere are projected onto the horizon circle or disk either as a lenticular figure (an area bounded by two congruent circular arcs) or as an arc of a circle. Both projections are evident in the figure for Proposition 9 in recension **a**, where lenticular figures are used to represent parallel circles and the ecliptic, and single arcs to represent the visible portions of the summer and winter tropics.

In the case of parallel circles, however, when the diagram concerns the largest circle bounding the northern stars that are always visible during the night at a given location (*scil.* the arctic circle), one must, strictly speaking, draw this circle tangent to the horizon circle; and in practice, it is often represented by such a circle inside the horizon. The antarctic circle (*scil.* the largest circle bounding the southern stars that are always invisible during the night at a given location) could equally well be represented by a small tangent circle inside the horizon circle and diametrically opposite the arctic circle. Clearly, diagrams involving the arctic, antarctic, and horizon circles would seem very cluttered. In such cases, the solution would be to represent the arctic and antarctic circles by small, diametrically opposed tangent circles outside the horizon circle.

Unfortunately, the individual drawings found in the manuscripts do not consistently adopt this solution of putting the invisible parts of the celestial sphere outside the horizon circle. For example, the first drawing for Proposition 2 [Menge 1916, 14] has two lenticular figures (KL and HN)

drawn wholly within the horizon circle to represent the whole ecliptic in two different positions. Were it not for inconsistencies like this, it would not be necessary to use (as we shall) dashed lines for arcs below the horizon, since such arcs should be represented by curves outside the horizon circle.[2] However, we have thought it better to to follow Menge and to forego strict consistency in drawing tangent circles in our figures. Often, however, when there are several large circles in a diagram involving tangent circles, we highlight the horizon by means of a weighted circle in order to make the figure easier to interpret.

In all cases, the direction of rotation of the sphere is right (eastern part of the horizon) to left (western part of the horizon)—which is why we stated above that the celestial sphere is viewed from the outside beyond the zenith.

[2] The diagram for Proposition 12 in **b** is noteworthy in that, even though the arctic circle does not appear in the diagram, arcs such as CX that are below the horizon are represented by arcs outside the horizon circle.

SIGLA

a	Menge's text based on Codex Vind. gr. 31.13
b	Menge's text based on Codex Vat. gr. 204
( )	punctuation of translation
[ ]	enclose interpolations into translation for clarity
< >	enclose interpolations into the Greek text as indicated by Menge

Euclid's *Phaenomena*
Translated with Commentary

Menge
1916

Euclid's *Phaenomena*[1]

[1]

[Introduction][2]

The fixed stars[3] are seen always rising from the same place and setting in [2]
the same place, those rising[4] simultaneously always rising simultaneously and those setting simultaneously always setting simultaneously. Moreover, they are seen as always having the same distances from one another as they move from rising to setting. Since this happens only with [things] that move in circular paths, when the eye [of the observer] is equidistant in every direction from the circumference, as is shown in the *Optics*,[5] one

[1] According to Menge [1916, 2], the manuscripts have the title, *Euclid's Phaenomena*. See English Glossary *s.v. Phaenomena*.

[2] Heath [1932, 96] calls this a preface.

[3] Since this treatise and, in particular, this sentence, are about their movement, it may seem strange to classify stars as fixed. The Greek means non-wandering and marks a contrast with those stars, that is, planets, that are said to wander because they rise at different points on the horizon and do not maintain the same angular distances in relation to the other stars.

[4] Rising and setting have to do with position with respect to the horizon (a term explained later by the author); they do not concern day and night, actual visibility and actual obscurity. See pp. 28–29, above.

[5] The *Optics* is, apparently, a treatise; but Neugebauer [1975, 756n13] states that the citation 'seems to have no basis in the extant works on optics.'

must postulate[6] that *the stars are carried*[7] *in circles and are set into one body*, and that *the eye is equidistant from the [circles'] circumferences.*

A certain star between the Ursae[8] is seen not to change from place to place but to revolve in the place where it is. And since this [star] is clearly equidistant in every direction from the circumferences of the circles on which the rest of the stars move, one must postulate that *all the circles are parallel*, so that all the fixed stars move on parallel [circles] having the aforesaid star as a pole.[9]

Some of these stars are seen neither to rise nor to set because they move on circles high in the sky, [circles] which they call ever-visible. These are the stars from the visible pole [extending] as far as the *arctic circle.*[10] And the stars nearer the pole move on smaller circles; those on the arctic circle, which in fact are seen to graze the horizon,[11] [move] on [the] largest [of these circles].

[4] All the stars to the south of these are seen both to rise and to set because their circles are not entirely above the earth[12] but part of them [is] above

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[6] The Greek text to this point is comprised in two subordinate clause, each beginning with 'since.'

[7] φέρεσθαι may be either be middle in voice (move oneself, move) or passive (be moved or carried).

[8] In Greek, ἄρκτοι, *arktoi* (Bears), whence later the term, 'arctic circle'; also called ἅμαξαι (Wains) [so, e.g., Aratus, *Phaenomena* vv. 26–27: cf. Heath 1932, 113].

[9] The pole is the celestial pole, usually specified by Euclid as the visible pole, and only once as the north pole in a suspect clause in the enunciation of Proposition 9.

[10] The arctic circle, a celestial circle, should not be confused with the geographical arctic circle at about $66\,1/2^{\circ}$N, the idea of which is nearly as ancient [Neugebauer 1975, 733]. Because of its dependence on the horizon, the celestial arctic circle depends on the terrestrial latitude of the observer: it becomes, for example, the (celestial) equator for an observer at either (terrestrial) pole, and shrinks to nothing when the observer is at the (terrestrial) equator. The celestial arctic circle is more commonly called 'the largest of the ever-visible circles,' which will be clearer than 'arctic circle' to modern readers at least. The designation, 'arctic circle,' occurs again after the following sentence in the Greek text, and is then ignored until Proposition 14. The relative absence of this term from the *Phaenomena* may already mark a stage of its disappearance from Greek astronomy, as mentioned by Neugebauer 1975, 582.

[11] The author uses the technical term 'horizon' in its non-technical sense here, before giving it its technical sense later in the Introduction.

[12] The expressions 'above the earth' and 'below the earth' are used consistently to mean above and below the horizon even after the horizon has been defined.

the earth and the remainder [is] below the earth. Of the segments of each of these [circles that are] above the earth, that [segment] seems larger which is nearer to the largest of the ever-visible circles, and of the segments below the earth, that [segment] seems smallest which is nearer to the same circle. [We know this] because the time of motion below the earth of the stars on this circle is least [that is, none] and [the time] of motion above the earth is greatest; and those [stars] on [circles] further off [from the visible pole] than these always have a lesser time of motion above the earth and a greater below the earth. Those [stars] nearest the south will have the least time of motion above the earth and the greatest below the earth. The stars on the [circle] in the middle of these [that is, halfway between the arctic and 'antarctic' circles] are seen to make [their] motion above the earth in a time equal to [that of their motion] below the earth: we therefore call this circle *equatorial*.[13] Those [stars that are carried] on circles equidistant from the equatorial circle make [their] motion in alternate segments in equal times; that is, those [segments] above the earth belonging to the northern[14] [stars have the same duration] as the [segments] below the earth belonging to the southern [stars] and the [segments] above the earth belonging to the southern [stars have the same duration] as the [segments] below the earth belonging to the northern [stars]. But the total time, that [spent] above the earth and the following [time] below the earth, is clearly the same for each circle.

Furthermore, the circle of the Milky Way[15] and the ecliptic,[16] which are oblique to the parallel circles and intersect each other, are seen always to have semicircles above the earth throughout [their] rotation.

On account of all the foregoing, then, let it be supposed that *the cosmos*[17] *is spherical*. For, were it either cylindrical or conical, the stars moving on [6] the oblique circles bisecting the equator would not always appear to move

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[13] The equator mentioned here is celestial not geographical; the English preserves a hint of 'equal', which is a part of the Greek original, 'equal-day circle'.

[14] Literally, 'toward the Bears'.

[15] The 'Milky Way', never mentioned again, is the 'milky circle' in the Greek, the word for milky, γάλακτος (*galaktos*), giving us our word 'galaxy'.

[16] 'Ecliptic', a term whose Greek equivalent is not used by Euclid, here and later translates ζῳδιακὸς (*scil.* κύκλος). In the remainder of the introduction to the treatise, other expressions for this circle will be translated literally; but, in the propositions, ζῳδιακὸς [κύκλος] (zodiacal circle) and ὁ τῶν ζῳδίων κύκλος (circle of the signs) will be translated by 'ecliptic'.

[17] We have not translated 'cosmos'. This word indicates everything enclosed by the sphere of the fixed stars, including the planets, the Sun, the Moon—none of which are ever mentioned—and the Earth.

on equal semicircles in [their] rotation but sometimes on a segment larger than a semicircle and sometimes on a [segment] smaller [than a semicircle]. For, if a cone or cylinder is cut by a plane not parallel to the base, the section arising is a section of an acute-angled cone,[18] which is like a shield. Now, it is clear that, when such a figure is cut through the centre along its length and along its width, dissimilar segments are made; and it is clear that, if it is cut in oblique sections through the centre, in this way also dissimilar segments are made, the very thing which does not appear to happen in the cosmos. For all these reasons then, the cosmos is spherical and turns uniformly[19] about its axis, one pole of which is above the earth, visible, and the other below the earth, invisible.

Let the plane through our eye, produced into the cosmos and bounding the portion seen above the earth be called [the] *horizon*;[20] [the horizon] is a circle, for if a sphere is cut by a plane the section is a circle.

Let a circle through the poles of the sphere and perpendicular[21] to the horizon be called a *meridian*[22] circle. And [let us call] *tropics* [the circles]

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[18] 'Section of an acute-angled cone' is a standard expression until Archimedes for an ellipse [cf. Heath 1921, ii 125]: the term, 'ellipse', is due to Apollonius. A section of a cone by a plane not parallel to the base is always an ellipse only if the cone is a right cone. Otherwise there are circular sections that can arise from certain positions of the cutting plane not parallel to the base: cf. Apollonius, *Conics* i 5.

[19] The fundamental property of uniform rotation of a sphere around its axis is stated later in this introduction in the argument that the ecliptic and equator bisect each other, and this same property is used as a definition at the beginning of *De sphaera*: see Mogenet 1950, 195. Neugebauer [1975, 756] points out that the content of this definition is later proved in *De sphaera* Theorem 2. Here and elsewhere in our translation and commentary, references to *De sphaera* are to Autolycus, *De sphaera quae movetur* (*On a moving sphere*) [Mogenet 1950].

[20] 'Horizon' now becomes a technical astronomical term, a plane disc, actually said to be the whole plane in the first clause, but limited in the second to being just a circle and ceasing to exist beyond the surface of the celestial sphere.

[21] Perpendicularity is the only specific angular relation between circles mentioned in this treatise.

[22] The meridian and tropics are astronomical, not geographical. Like 'equator', these terms should not cause confusion, even though they designate positions on the celestial sphere corresponding to the positions of their geographical counterparts on a terrestrial globe under central projection. On some modern globes, there is marked a circle that corresponds in the same way to some fixed position of the ecliptic.

that have the same poles as the sphere and that the circle through the middle of the signs [the ecliptic] touches.[23]

The circle through the middle of the signs and the equator are great [circles], for they bisect each other. For, the beginning of Aries and the beginning of Libra are diametrically opposite; and, because they are on the equator, they rise and set in syzygy,[24] having between them six of the twelve signs of the ecliptic and two semicircles of the equator, since the [8]
beginning [of Aries and Libra], because it is on the equator, moves in the same time, the one moving above the earth and the other below the earth. But, if a sphere turns uniformly about its own axis, all the points on the surface of the sphere traverse similar arcs[25] of the parallel circles on which they move in equal times. Therefore, [the beginnings of Aries and Libra] will traverse similar arcs of the equator, the one above the earth and the other below the earth. Therefore, the arcs are equal. Therefore, each is a semicircle: for, from rising to rising or from setting to setting is a whole circle. Therefore, the circle of the signs and the equator bisect each other. But, if two circles in a sphere bisect each other, each of them is a great circle: therefore, the circle of the signs and the equator are great [circles].

The horizon too is one of the great circles: for, it always bisects both the circle of the signs and the equator, which are great [circles]. For, six of the twelve signs are always above the earth, and a semicircle of the equator is always on high. For, the fact is that the stars on this [circle] that rise and set at the same time [that is, are diametrically opposed] come in the same times, the one from rising to setting and the other from setting to rising. Now, from what has been shown previously, it is evident that a semicircle of the equator is always above the horizon. But, if a circle that remains fixed in a sphere bisects any great circle that is always moving, the cutting [circle] is also a great [circle]. Therefore, the horizon is a great circle.

The *time of revolution* of the cosmos is the [time] in which each of the fixed stars comes from a rising to the next rising or from an arbitrary place [10]
to the same place.

An arc's *passage across*[26] the visible hemisphere occurs while the leading point of the arc rises, the trailing point rises, traverses the whole hemi-

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[23] The noun, τροπή (*tropē*), translated by 'turn' in the enunciation of Proposition 7, is a standard term for a solstice. τροπικόι [*scil.* κύκλοι] (tropical circles) is here rendered by 'tropics'.

[24] In Euclid's time, this term was not restricted to the conjunctions and oppositions of the Sun and Moon.

[25] 'Similar' is a technical term: cf. English Glossary *s.v.*

[26] Euclid uses the notion of an arc's passage across a hemisphere in Propositions 14 to 18. The term used for it here is peculiar to Euclid and Theodosius,

sphere above the earth, and sets: and [its passage across] the invisible [hemisphere occurs] while the leading point of the arc sets, the trailing point sets, traverses the whole hemisphere below the earth, and rises.

Since we have earlier explained our reasons for doubting the authenticity of the introduction, we shall refer in our commentary to its author by some such circumlocution as 'the author,' 'the writer,' and so on.

In his introduction, the author presents arguments for certain assumptions about unobservable features of the cosmos based on common celestial phenomena. The words used to introduce these assumptions are, in the first two instances, forms of the verb, τίθημι (suppose or posit); and, in the third, it is a form of ὑπόκειμαι (be supposed or posited). In contrast, in his *Elements*, Euclid calls his famous five Postulates (αἰτήματα). And, unlike the *Elements* where no arguments are given for accepting the postulates, the introduction to this treatise seeks to justify these postulates by an appeal to the following observable phenomena:[27] any given star always rises and sets in the same places and two stars not only always appear to have the same distance from each other but, if they rise or set together once, they always do so. These facts, he says, induce us to postulate that:

(1) the stars move in circles,
(2) they are affixed to the surface of a single sphere and are everywhere equidistant from the eye,
(3) the circles are parallel, and
(4) all these parallel circles have a single star between the Ursae as their pole.

In fact, in Euclid's time there was no such star. T. L. Heath [1932, 119] quotes Hipparchus, *In Arat. et Eud. Phaen.* i 4.1, on the subject of the pole star. Hipparchus says that the astronomer Pytheas of Massalia, whose *floruit* of 320 BC makes him a probable contemporary of Euclid, pointed out that the place of the north celestial pole is not occupied by a star but is the fourth corner of a square whose other corners are α and κ of

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and means literally 'interchange': cf. Neugebauer 1975, 759-760. This translation of ἐξαλλαγή was was chosen to fit the translation of the corresponding verb (ἐξαλάττω), which appears in the propositions.

[27] One recalls Aristotle's distinction in his *Analytica posteriora* between derivation of first principles from the phenomena and the derivation of phenomena from first principles. Both types of derivation are found within the *Phaenomena*, the former in the introduction and the latter in the body of the treatise.

Draco and β of Ursa Minor.[28] In the same passage, Hipparchus criticizes Eudoxus' error in asserting that, in Heath's translation, 'There is a certain star which remains always in the same spot,' wording reminiscent of that in the introduction to the *Phaenomena*.

Of these parallel circles some are ever-visible, the largest of which the writer calls—as does Geminus[29]—the arctic circle. South of this come the circles parallel to it, which either cut the horizon and so have each a visible and an invisible segment, or are completely below the horizon and are, therefore, entirely invisible.

The nearer a parallel circle is to the arctic circle the larger its visible segment and the smaller its invisible: for, so the text says, stars on these circles are observed to take more or less time to cross the sky according as the circles on which they are carried are nearer to or further from the arctic circle (provided they are to the south of it, of course). Moreover, stars on the middle of these parallel circles, the equator, take equal times above and below the horizon; and the time it takes a star to traverse the visible segment of a parallel circle on one side of the equator is equal to the time it takes a star to traverse the invisible segment of the parallel circle equidistant from the equator on the other side. And the combined times that it takes a given star to traverse the visible and invisible segments of its circle is the same as the combined times for any other star.

All of this, apart from the previous sentence, is equivalent to the Visibility Theorem, stated in our account of Euclid's presuppositions [see p. 24, above], provided that one also assumes (as the writer does explicitly later in the introduction) that the stars are carried uniformly on their circles. However, the Visibility Theorem uses an exclusively geometric vocabulary of circles and arcs, whereas the writer of the introduction begins by relating comparisons of arcs to comparisons of times and then compares times for the rest of the argument.

Considerations of parallel circles and visible and invisible arcs, necessary in themselves, are also put to use in establishing the shape of the cosmos once the writer has noted that, in addition to the foregoing, both the ecliptic and the Milky Way[30] appear to be bisected by the horizon at all

[28] Indeed, a glance at the orbit of precession of the north pole [see SC002 Constellation Chart: Northern Circumpolar Region—Epoch 2000] shows that during the Hellenistic period there was no star of magnitude 6 or less that could possibly have functioned as a pole star for an observer in the northern hemisphere.

[29] According to Neugebauer [1975, 580–581], Geminus was active in the first century AD.

[30] Neither of these terms is explained. Since the Milky Way is never mentioned again, this may not matter so far as it is concerned; but the ecliptic plays a

times. Because of all these things, then, he says, the cosmos is assumed to be spherical.

The drift of the subsequent argument for this conclusion is as follows. His opening remarks about the revolution of the cosmos around an axis establish the cosmos as a solid of revolution, which, without any explanation of why, is tacitly assumed to be, if not a sphere, then either a right cone or a right cylinder.[31] Now, the horizon plane is tacitly taken as one of the planes 'not parallel to the base' of such a figure, which is to say that the observer is not at a pole; so, the horizon cuts the cosmos in an ellipse (referred to not as such but as 'a section of an acute-angled cone, which is like a shield').[32] Similarly, any other rotating section of the solid, such as the ecliptic or Milky Way, that is also skew to the solid's unique family of parallel circles will intersect the portion of the solid above the horizon in quite dissimilar figures during the course of one rotation, since these too will be segments of ellipses. But this is contrary to the observed facts about the ecliptic and Milky Way reported earlier. Therefore, the cosmos cannot be cylindrical or conical.

With this the preliminary part of the introduction is complete. The cosmos has now been shown to be spherical; and the next step is to present the important sections of this sphere. Hence, there follows, first, the definition of the horizon, as the section of the sphere produced by the plane through the eye and bounding the visible portion of the cosmos. Unlike the ecliptic or equator, it does not rotate with the sphere, since it is defined by an observer. In terms of the earlier definition of the equator, this implies that the equator is always bisected by the horizon.

Next, the meridian is defined as the circle through the poles of the sphere perpendicular to the horizon, and the tropics as the parallel circles that the ecliptic always touches. (To show the existence and uniqueness of the meridian it would be necessary to invoke the Perpendicularity Theorem.)

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major role in the treatise, and one wonders why it is not defined. One reason may be that the ecliptic, unlike the horizon and equator, is not a mathematical consequence of the model (an observer on a spherical Earth of minute size in relation to a uniformly rotating spherical heaven) but simply a physical feature of reality, whose principal feature of relevance to the introduction is that it always bisects the horizon and the equator.

[31] These same solids appear as possible shapes to be disposed of in *Alm.* i 3, but as possible shapes for the Earth, not of the cosmos as a whole.

[32] We have already remarked that the phrase 'section of an acute-angled cone' is prior to Apollonius; but the added phrase 'which is like a shield' occurs as a description of an ellipse in the *In prim. Euc.* [Friedlein 1873, 103.7] of Proclus, who lived in the fifth century of our era.

The writer then states that the ecliptic and equator are great circles, which he attempts to prove by showing that they bisect each other. The argument starts with the simple assertion that the beginnings of Aries and Libra lie on the equator; so, since they contain (between them) six (of the twelve equal) signs of the ecliptic, the equator bisects the ecliptic at these points. But this means that these points are diametrically opposite on the ecliptic; thus, they rise and set in opposition (which the author calls 'in syzygy'). Now, the author cites a theorem about a sphere turning uniformly on its axis and concludes that the beginnings of Aries and Libra traverse similar arcs of the equator from setting to rising and from rising to setting. Therefore, since these are arcs of one circle, they are equal arcs; consequently, the arcs of the equator separating the beginnings of Aries and Libra are equal to each other, that is, the ecliptic bisects the equator. Hence, since the ecliptic and the equator bisect each other, they are great circles.[33]

Next it is stated that the horizon is also a great circle, something the writer seems to want to prove twice. The first proof is based on the fact that the horizon always bisects the ecliptic (for which he appeals to the observation that six signs of the ecliptic are always above the earth); and the second, on the fact that the horizon always bisects the equator. (To establish this latter fact, he reminds us of his previous argument concerning beginnings of Aries and Libra and says that it can be generalized to prove that, of any two simultaneously rising and setting stars, if one rises as the other sets, then we later see the former set as the latter rises. Thus, the visible arc of the equator is traversed in the same time as the invisible; accordingly, they are equal arcs and, therefore, semicircles.)

A problem with the proof given is that the equator is defined earlier as that parallel circle with equal visible and invisible parts. So, given the definition of horizon earlier as that plane separating the visible from the invisible, it is a consequence of these definitions that the horizon bisects the equator. In effect, the section simply repeats of what was said earlier.

[33] Neugebauer [1975, 756], in referring to 'proofs for the greatcircle [*sic*] quality of the ecliptic, equator and horizon (based on mutual bisecting)' says that 'Most of this material does not fit the text [of the body of the treatise] or is repeated in it.' He does not, however, say where it is repeated; and we have been unable to find the repetition. In any case,'most of this material' surely exaggerates the amount of repetition. As for the material's fitting the text, we note, as an example of a fit, the implicit use in Proposition 1 of the fact that the horizon and ecliptic bisect each other. (See our commentary on that proposition.) We do not, however, deny a certain amount of repetition within the introduction itself, an example being the putative proof, just prior to the end of the introduction, that the horizon is a great circle, a thesis we discuss below.

In any case, having established that the horizon always bisects both the ecliptic and equator, the writer now cites the following theorem:

> If a fixed circle always bisects a moving great circle, then the fixed circle is great.

Since he has shown that the fixed circle of the horizon always bisects the ecliptic and equator, it now follows that the horizon is a great circle.

Of course, 'always' and 'moving' are unnecessary in this theorem, since it is true that, 'if a given circle on the sphere bisects a great circle, then the given circle is great.' This alone, when applied to the case of the horizon and either the ecliptic or equator, would suffice to show that the horizon is a great circle.

The theorem cited in the text bears a superficial resemblance to *De sphaera* Theorem 12; and, indeed, Menge glosses this theorem here. One should note, however, that Autolycus does not assume that either the fixed circle or the moving circle is known to be great, although he does require other assumptions; whereas the theorem just quoted above supposes that the moving circle is known to be great, which obviates all need for it to be moving.

The introduction concludes with two definitions, namely, that of the time of revolution of the cosmos[34] and that of the passage of an arc across the visible (invisible) hemisphere, the latter being reckoned from the appearance (disappearance) of the leading point to the disappearance (appearance) of the trailing point. The significance of this latter notion is that, although from a mathematical standpoint the rising time of the semicircle of the ecliptic following a point P can be looked upon as the length of daylight when the Sun is at P, it is in fact only a close approximation. For, the length of daylight is precisely the time it takes the arc of the ecliptic occupied by the disk of the Sun at sunrise, together with the arc traversed by the trailing edge of the disk[35] during the day, to cross the visible hemisphere.

[Proposition] 1. *The Earth is in the middle of the cosmos and occupies the position of centre with respect to the cosmos.*

In the cosmos [Figure 9], let AB be a horizon, [let] the Earth ([that is,] our eye) [be] at point D; let the eastern side be G and the western [be] A; and let Cancer be observed rising at point G with a diopter placed at point

[34] Cod. Vat. gr. 204 omits this definition, and Euclid never refers to it.

[35] This is a small arc (about a half-degree) due to the annual motion of the Sun.

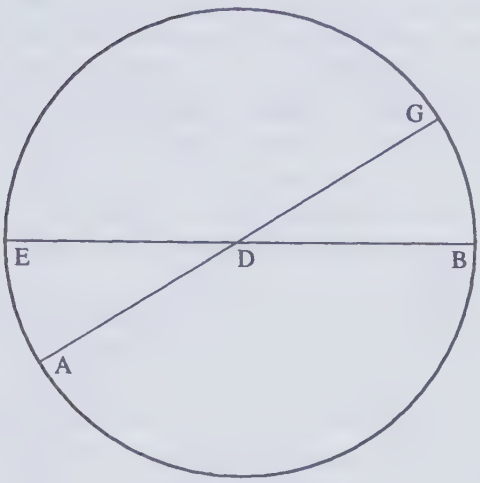

Figure 9. Proposition 1

D. Then, with the same diopter [in the opposite direction] Capricorn will be observed setting. Let it be observed [setting] at point *A*. Since points *A*, *D*, [and] *G* have been observed with a diopter, there is a straight [line] through *A*, *D*, [and] *G*. Hence, *ADG* is a diameter of the sphere of the fixed [stars] and [a diameter] of the ecliptic, since it cuts off six signs above the horizon.

Then, again, after a movement of the ecliptic and of the diopter, let Leo [12] be observed rising at point *B*; then, with the same diopter Aquarius will be observed setting. Let it be observed at point *E*. Since the points *E*, *D*, [and] *B* have been observed with a diopter, there is a straight [line] through *E*, *D*, [and] *B*. Let it be *EDB*.[36] The [line] *EDB* is, therefore, a diameter of the sphere of the fixed [stars] and of the ecliptic. But *ADG* was shown [to be a diameter] also; therefore, the point *D* is the centre of the sphere of the fixed [stars], and it is at the Earth.

Similarly we shall show that, whatever point be chosen on the Earth, it is the centre of the cosmos.

Therefore, the Earth is in the middle of the cosmos and occupies the position of centre with respect to the cosmos.

Paraphrase. The Earth is at the centre of the cosmos.

[36] Here and immediately following are two of the very many places where the Greek indicates what the letters represent by means of the gender and number of the definite article.

Remark. This proposition is unusual in that it is the only one in the treatise to state a physical feature of the two-sphere model; all the other propositions state mathematical features of the model.

The proof, which is unique among among the propositions in this treatise because it appeals to imagined observations, has three parts:

(1) any observer is situated on two diameters of the ecliptic—note the appeals to imagined observations with the diopter—accordingly, because the ecliptic is a great circle, the observer is situated on two diameters of the cosmos,

(2) therefore, any observer is situated at the centre of the cosmos, and

(3) since this is true for any observer, the Earth itself must lie at the centre of the cosmos.

An unstated consequence of the proof is that the size of the Earth is negligible relative to that of the cosmos, a notion which was a commonplace of Hellenistic astronomy, and which we discuss elsewhere.

The diopter to which Euclid appeals in the proof was an ancient theodolite used for both astronomical and surveying purposes. Its construction and use were subjects of the treatise *De dioptra* by Heron of Alexandria, extracts of which (together with a picture of the reconstructed diopter) may be found in Cohen and Drabkin 1948, 139 and 336–342. The reader is to recall that two diameters of the ecliptic, the one through the beginnings of Cancer and Capricorn and the other through the beginnings of Leo and Aquarius, pass through the viewing piece of the diopter; hence, the observer is at the centre of the ecliptic.

To go from this to the conclusion that the observer is at the centre of the cosmos, one may regard Euclid's singling out two pairs of opposite points on the ecliptic as a reminder about what would be observed for any pair of opposite points. One might then conclude that a fixed circle, the horizon, bisects a moving circle, the ecliptic, neither of them being perpendicular to the axis or containing the poles of the parallels. It would then follow from Theorem 12 of *De sphaera* that both the horizon and ecliptic are great circles. And, hence, an observer at the centre of the ecliptic would also be at the centre of the cosmos.

The difficulty in this interpretation is Euclid's conclusion, stated after each setting of the diopter in the proof, that, since the observer sights diametrically opposite points of the ecliptic through the diopter, he is on a diameter of the sphere of the fixed stars, an argument which assumes the ecliptic is a great circle and so (apparently) presupposes the argument in the introduction that the ecliptic is a great circle in the cosmos. Given our view that the introduction as it stands was not part of the original treatise, we assume either that such an argument was intended to fill a gap

in this proof, or that Euclid refers here to a postulate that may have been one of a few postulates at the beginning of the treatise which later were incorporated into the fuller introduction of the present text.

Finally, since the above arguments hold wherever the diopter is located on the Earth, the Earth itself is at the centre of the cosmos.

[Proposition] 2. *In one revolution of the cosmos, the circle through the poles of the sphere will be perpendicular to the horizon twice; and the ecliptic will be perpendicular to the meridian twice but never to the horizon, when*[37] *the zenith*[38] *is between the summer tropic*[39] *and the visible pole.* <*If the zenith is on either of the tropics, the ecliptic will be perpendicular to the horizon once; and, when the zenith lies between the tropics, the ecliptic will be perpendicular to the horizon twice.*>

Let EBG be a horizon [Figure 10]; let WD be the largest of the ever-visible circles; let EZ be the largest of the ever-invisible circles; HQK, the summer tropic; LMN, the winter tropic; and let the ecliptic have position[40] KL, and points X [and] O [be] poles of the sphere. Let there be drawn through X [and] O a great circle $AXEOP$. [14]

[*Part 1*]. I say that, in one revolution of the sphere, the circle through the poles of the sphere will be perpendicular to the horizon twice and the ecliptic will be perpendicular to the meridian twice but never perpendicular to the horizon, when the zenith lies between the circle HQK and the pole X.

Now, it has been shown [elsewhere] that the [circle] through the poles of the sphere is perpendicular to BEG twice.[41]

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[37] 'When' means 'where on the surface of the Earth'.

[38] Euclid always refers to the visible pole of the horizon (in modern parlance, 'the zenith') as 'the pole of the horizon', as though it had only one, even when in the same sentence he distinguishes the visible pole of the parallel circles.

[39] The summer tropic (literal translation) is the tropic at which the Sun turns at the summer solstice, similarly the winter tropic [cf. p. 47]. The designations, 'tropic of Cancer' and 'tropic of Capricorn', are not found in Euclid's text.

[40] Euclid almost always remembers to specify the 'position' of the ecliptic, rather than saying just that KL is the ecliptic, in order to emphasize that the ecliptic moves in relation to the other parts of the configuration.

[41] 'Elsewhere' is Neugebauer's inference from δέδεικται (it has been shown) [1975, 750]. He attributes to this remark half the argument of some scholars that *De sphaera* predates the *Phaenomena* because the theorem and proof appear as Theorem 10 in the former. (The other half of the argument concerns the inter-

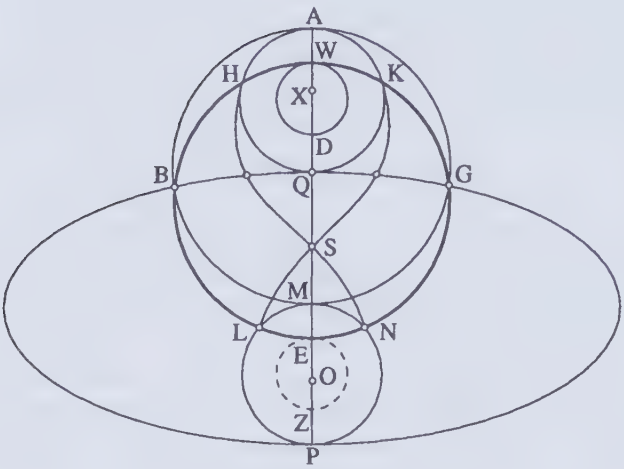

Figure 10. Proposition 2

(This is a copy of the figure in the Vatican manuscript: whoever drew the figure in cod. Vind. gr. 31.13 represented two distinct parallel circles on the sphere—the arctic circle and the summer tropic—as having a common point.)

[16] [*Part 2.1*]. I say also that KL will be perpendicular to the meridian AQ twice.

[*Proof*]. Since in the sphere the two circles WBG [and] HQK intersect each other, and [since] the great circle AQO is drawn through their poles, arc HQ is equal to [arc] QK; and, for the same [reasons], arc LP [is equal] to arc PN. Arc HQK is also equal to arc LPN. Therefore, arc LP is equal and similar to arc KQ. Therefore, while point K (beginning from point K) traverses arc KQ and comes to Q, [point] L traverses arc LP and comes to P, and the ecliptic will have position $QBPG$. Now, since in the sphere the two circles HQK [and] $QBPG$ touch each other, and the great circle $XQOP$ has been drawn through the poles of one and the point of contact [of the two], $XQOP$ will also pass through the poles of $QBPG$ and will be perpendicular to it. Hence, $QBPG$ is perpendicular to $XQOP$.

Again, since arc QH is similar to arc PN, when Q comes to H, P [will come] to N, and the ecliptic will have position HSN. Moreover, since arc

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pretation of the proof of Proposition 3 as dependent on Autolycus.) Neugebauer assumes reasonably that this sentence is not a specific reference to Autolycus, but a later reference to generally known theorems that has replaced the first of the expected proofs, 'perhaps in a damaged copy.'

HA is [equal and] similar to arc MN, when H comes to A, N [will come] to M, and the ecliptic will have position $ABMG$. Now, since in the sphere the two circles $ABMG$ [and] AQ touch each other, and the great circle $AXQO$ has been drawn through the poles of one and the point of contact [of the two], $AXQO$ is perpendicular to $ABMG$. Hence, $ABMG$ is also perpendicular to $AXQO$.

Again, since arc AK is [equal and] similar to arc LM, when A comes [18]
to K, M will come to L, and the ecliptic will have position KL. Therefore, within the time in which K (beginning from K) traverses the arc $KQHAK$ and reaches K (which is the time of one revolution of the cosmos), circle KL will be perpendicular to circle AO twice.

[*Part 2.2*]. On the same assumptions, let the pole of circle BEG be between the points Q [and] X.

I say that the ecliptic KL will never be perpendicular to the horizon BEG.

[*Proof*]. If the [circle] KL were perpendicular to $BEGK$, it would pass through its poles and through the pole of the horizon, which is between [20]
points Q [and] X; and it would cut AHK, the very thing it cannot do. Therefore, the ecliptic KL will never be perpendicular to the horizon $BEGK$.

<[*Part 3.* Figure 11]. Let the zenith be point M on circle $AHMK$.

I say that the ecliptic will be perpendicular to the horizon once.

[*Proof*]. Since arc KM is similar to arc LN, while K traverses arc KM and comes to M, L will come to N, and the ecliptic will have position $MBNG$. Now, since $MBNG$ intersects HBG [and passes] through its poles, it bisects [HBG] perpendicularly. Therefore, the ecliptic is perpendicular to [22]
the horizon.

[*Part 4.* Figure 12]. Let the zenith be point O between the tropics.

I say that circle KL will be perpendicular to the horizon twice.

Let the great circles SOT [and] POR be drawn through the zenith O touching the circle $AHMK$; then, they will also touch [circle] NR. And, since POR passes through the poles of HEK, it bisects [HEK] perpendicularly. Therefore, circle PR is perpendicular to [circle] HEK.

For the same [reasons], [circle] SOT is perpendicular to [circle] HEK. And, since the semicircle from K in the direction from K to L is asymptotic[42] to the semicircle from S in the direction from S to T, arc KS is similar to arc LT. Therefore, when K comes to S, L will come to T, and circle KL will coincide with circle ST. But circle ST is perpendicular to [circle] HEK; and, therefore, circle KL is perpendicular to circle HEK.

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[42] See p. 60, below.

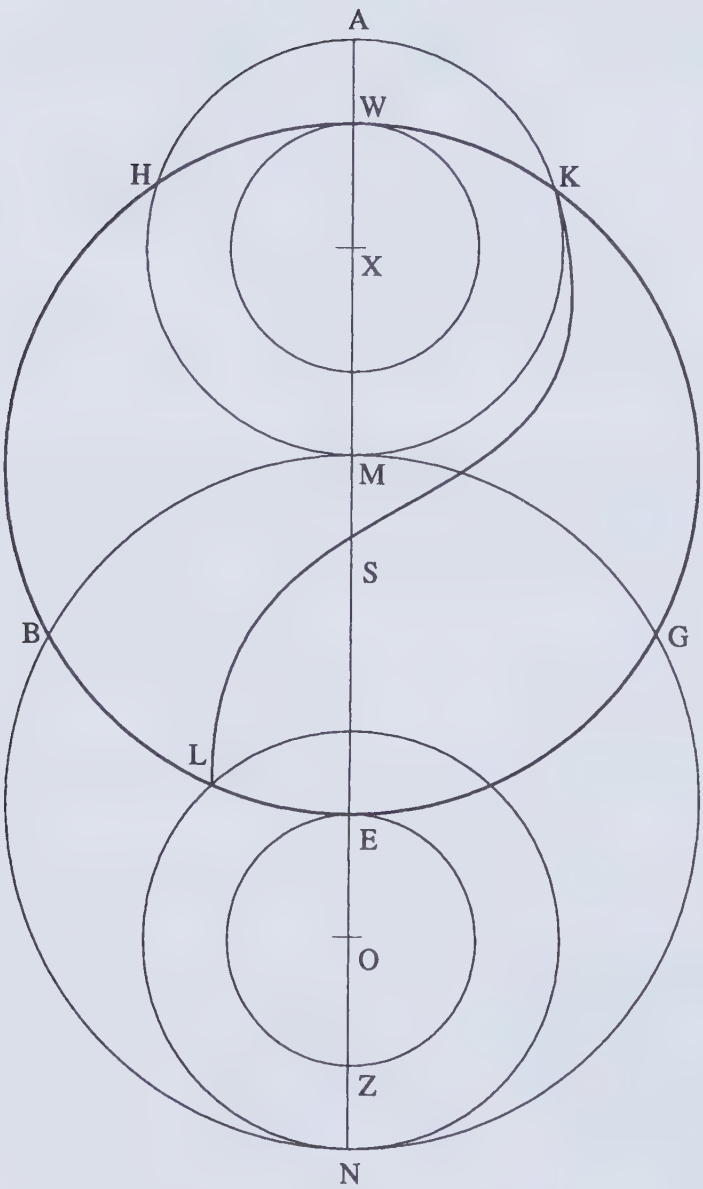


Figure 11. Proposition 2. Part 3

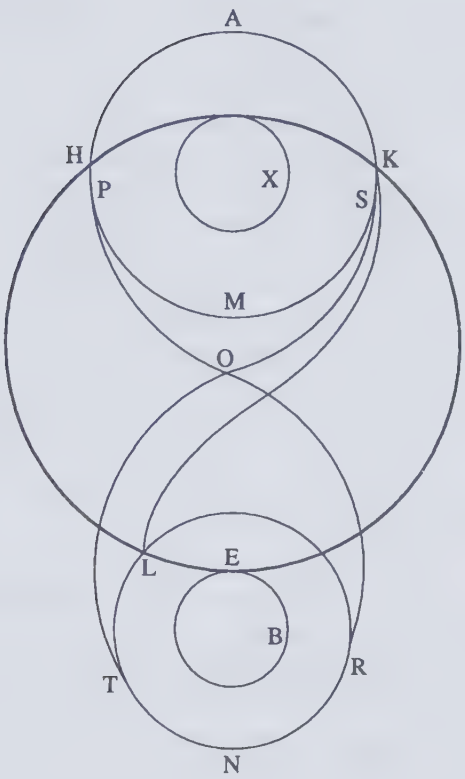
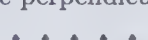


Figure 12. Proposition 2. Part 4

Again, since arc SMP is similar to arc TNR, when S comes to P, T will reach R and the ecliptic will coincide with the circle POR. But circle POR is perpendicular to HEK; and, therefore, the ecliptic is perpendicular to the horizon, HEK.

Therefore, the ecliptic will be perpendicular to the horizon twice.>



Paraphrase. While the heavens turn once around us, any circle through the poles of the sphere that rotates with the heavens will be perpendicular to the horizon twice; and the ecliptic will be perpendicular to the meridian twice, but never to the horizon, if the zenith is outside of the tropics. <If it is on a tropic, the ecliptic will be perpendicular to the horizon once; and, if it lies between the tropics, then the ecliptic will be perpendicular to the horizon twice.>

Proof. Euclid says that the first statement has already been proved; and indeed it was in *De sphaera* Theorem 10. Since the definition of the meridian is that it is the (unique) circle through the poles of the sphere perpendicular to the horizon, the force of the first part of Proposition 2 is that a given circle through the poles of the sphere twice coincides with the meridian.

In the proof of the second statement Euclid argues that the ecliptic, which is always tangent to the tropics, passes through the intersections of both tropics with the meridian simultaneously. But, when it does this, the meridian, which contains the poles of the tropics, will pass through the poles of the ecliptic, by the Tangency Theorem. And, then, by the Perpendicularity Theorem, the meridian and ecliptic will be perpendicular. Euclid says nothing about why the ecliptic is perpendicular to the meridian only at these two times. However, such a proof would evidently not involve kinematic considerations, but would rely on the Perpendicularity Theorem to establish that,

(1) when it is perpendicular to the meridian, the ecliptic passes through the poles of the meridian;

and then it would show that

(2) this happens only when the ecliptic passes through a point where the meridian and tropics intersect.

Euclid's evident lack of interest in demonstrating this probably reflects his interest in only those features that involve motion.

Given the Perpendicularity Theorem, the ecliptic is never perpendicular to the horizon, because, the poles of the horizon would have to lie in a region where the ecliptic could pass through them, that is, within the tropics, which is specifically excluded. Once this is understood, it is evident how the remaining, interpolated, portions are to be proved.

The phrase in Part 4 about one semicircle's not meeting another means that they intersect only at their centres and not 'where it counts', that is, on their circumferences. This is the significance of the claim that they are asymptotic. (According to Neugebauer [1975, 758], ἀσύμπτωτος (asymptotic) received its modern meaning in Apollonius' *Conics*.)

[24] [Proposition] 3. *Of the fixed stars that rise and set, each [always] rises and sets at the same points of the horizon.*

In the cosmos [Figure 13], let ABG be the horizon; let circle ADE be the largest of the ever-visible circles; and BHZ, the largest of the ever-invisible [circles]. Let a star Q be chosen [to be] one of the stars that rise and set, and let G be the eastern side and K the western.

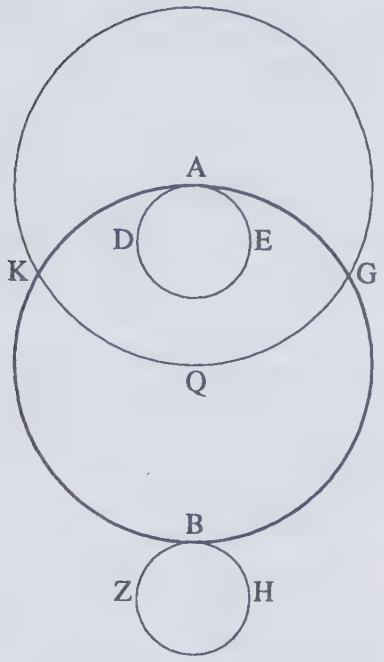


Figure 13. Proposition 3

I say that the point Q always rises and sets at the same points of the horizon in a rotation of the sphere.

Let KQG be the circle on which point Q moves; then, circle KQG intersects the horizon and is perpendicular to the axis of the sphere. But circles perpendicular to the axis and intersecting the horizon have risings and settings [always] at the same points of the horizon. Therefore, circle KQG always rises at the point G and sets at K. And the star Q moves on the circumference of the circle KQG; therefore, the star Q always rises at the point G and sets at K.

Paraphrase. A star that rises and sets does so always at the same two points of the horizon.

Remark. This proposition proves the remark offered in the introduction as a reason why stars must be carried on circles, namely, that a given star always rises and sets at the same place on the horizon. This deduction from basic premises of what was earlier stated as an argument for these premises suggests that the introduction is an attempt to go from the celes-

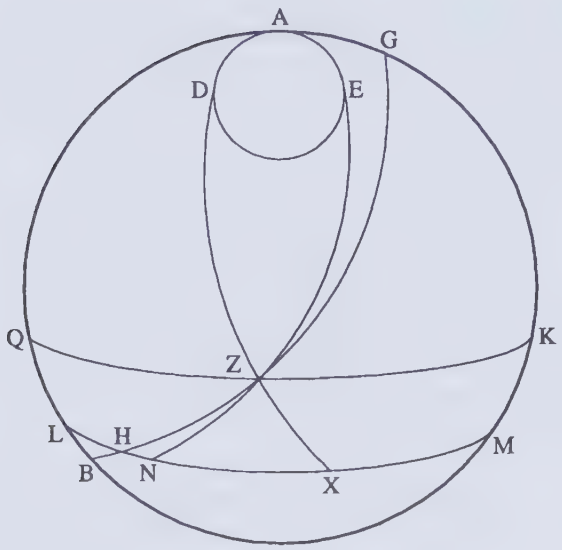


Figure 14. Proposition 4

tial phenomena to a few fundamental principles which Euclid shows suffice to imply the same phenomena [cf. pp. 11–12, above].

Proof. A star not on the axis of the sphere is, by *De sphaera* Theorem 1, carried on a circle perpendicular to the axis. By *De sphaera* Theorem 7, such circles always cross the horizon at the same points; that is, they do not wobble, as the last statement in Proposition 7 says the ecliptic does. Thus, the star that is carried on this circle always crosses the horizon at the same points as it rises and sets.

[Proposition] 4. *Of the stars on the circumference of a great circle that neither intersects the largest of the ever-visible circles nor touches it, those rising earlier set earlier, and those setting earlier rise earlier.*

[26] In the cosmos [Figure 14], let ABG be the horizon; ADE, the largest of the ever-visible [circles]; let GZB be a different great circle (neither intersecting nor touching ADE); and let two arbitrary points Z [and] H be chosen on the circumference of GZB.

I say that, of the points Z [and] H, [Part 1] the earlier to rise sets earlier and [Part 2] the earlier to set rises earlier.

[*Part 1.1*]. Let G be the eastern side and B the western, and let QK [and] LM be the parallel circles on which points Z [and] H move, and let

there be drawn through Z a great circle NZE touching the circle ADE so that the semicircle from E in the direction of EZN is asymptotic to the semicircle from A in the direction of AG. Then, arc KZ is similar to arc MN. Therefore, the remaining arc ZQ and the [arc] adjacent to it below the earth as far as point K are [together] similar to arc NL and the [arc] adjacent to it below the earth as far as point M. Therefore, points Z [and] N traverse ZQ [and] NL and the [arcs] adjacent to them as far as points K [and] M in an equal time. Therefore, points Z [and] N rise at the same time, but H rises earlier than N, and therefore H rises earlier than Z.

[*Part 1.2*]. I say that it [H] also sets earlier.

Let there be drawn through point Z the great circle XZD touching circle ADE and such that the semicircle from D in the direction of DZX is asymptotic to the semicircle from A in the direction of AQ. Therefore, arc ZQ is similar to arc XL; therefore,point Z traverses arc ZQ and point [28]
X [traverses] arc XL in an equal time. Therefore, when point Z comes to point Q, [point] X also comes to L. Therefore, the points Z [and] X set simultaneously. But H sets earlier than X; therefore, H also sets earlier than Z.

[*Part 2*]. Similarly, we shall show that the earlier to set also rises earlier.

Paraphrase. If all points on the circumference of a certain great circle rise and set, then stars on it that rise earlier set earlier, and *vice versa*.

Remark. This proposition has its principal applications in Propositions 9, 14 and 15, where the 'great circle' in the hypothesis is the ecliptic. To say that all parts of the ecliptic rise and set is the same as saying that the Sun rises and sets each day, that is, one does not see the so-called midnight Sun.

Proof. Let Z and H be two stars and GZH the great circle. Suppose that ABG is the horizon, with B being the western part and G the eastern part; and let ADE be the greatest ever-visible circle. Let QK and LM be the circles on which the stars Z and H are carried. Let EZN be the initial arc of the semicircle through Z and tangent to ADE that does not intersect the semicircle of the horizon beginning with the arc AKM [cf. Tangent Circle Theorem]. By the Asymptotic Semicircles Theorem, the arcs KZ and MN are similar, so that the remaining arcs of the two circles ZQK and NAM are similar. Thus, N and Z will rise simultaneously.

To prove Part 1, suppose that H rises before N, so that it will rise before Z. To show that it also sets before Z, draw through Z the great circle XZD tangent to the arctic circle at D so that the semicircle in the direction DZX does not intersect the western horizon. By the Asymptotic Semicircles Theorem, arc ZQ is similar to arc XL; thus, Z traverses ZQ

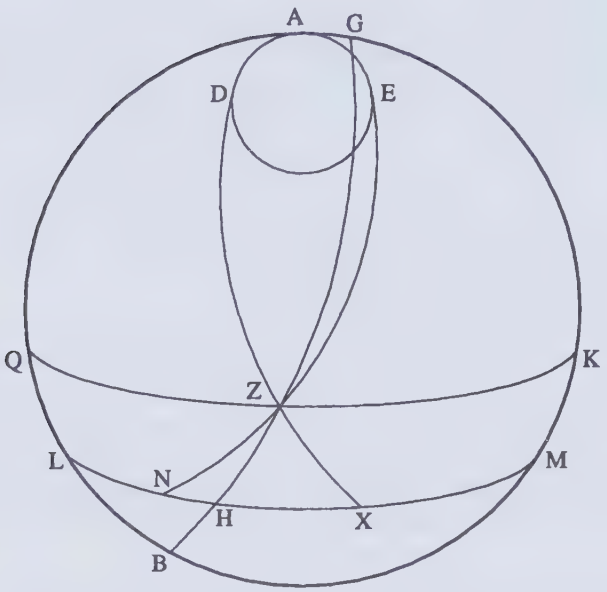


Figure 15. Proposition 5

and X arc XL simultaneously. Hence, Z reaches Q, that is, Z sets when X reaches L, that is, when X sets. Since Z and X set at the same time but H sets before X, H also sets before Z.

As Euclid says, the proof of Part 2 would be similar. Curiously, Euclid does not prove the second by geometrical conversion, as he proves so many converses in the *Elements*. Perhaps this is because, as Menge points out, he avoids proof by contradiction in the *Phaenomena*.

[Proposition] 5. *Of the stars on the circumference of a great circle that intersects the largest of the ever-visible circles, those toward the north both rise earlier and set later.*

In the cosmos [Figure 15], let ABG be the horizon; ADE, the largest of the ever-visible [circles]; and let GZB be a different great circle intersecting the circle ADE. Let two arbitrary points Z [and] H be chosen on the circumference of GZB, and let the point Z be toward the north [from H]. [*Part 1*]. I say that the point Z both rises earlier than H and sets later.

Let G be the eastern side and B the western; let the parallel circles on which points Z [and] H move be QK [and] LM; and let there be drawn

through point Z a great circle NZE touching circle ADE such that the semicircle from E in the direction of ZN is asymptotic to the semicircle from A in the direction of AGK [cf. Tangent Circle Theorem]. [30]

Then, arc ZK is similar to arc NM; therefore, the remaining arc QZ and the [arc] adjacent to it below the earth as far as point K [together] are similar to arc LN and the [arc] adjacent to it below the earth as far as point M. Therefore, points Z [and] N traverse the arcs ZQ [and] NL and the [arcs] adjacent to them as far as the points K [and] M in an equal time. Therefore, points Z [and] N rise simultaneously, but N rises before H; and, therefore, Z rises before H.

[*Part 2*]. I say that it [Z] also sets later.

Let there be drawn through point Z a great circle XZD touching circle ADE such that the semicircle from D in the direction of DX is asymptotic to the semicircle from A in the direction of AQ.

Then, arc ZQ is similar to arc XL. Therefore, Z traverses arc ZQ and X [traverses arc] XL in an equal time. Therefore, when Z reaches the point Q, [point] X reaches L. Therefore, the points Z [and] X set at the same time, but H sets earlier than X; and, therefore, H sets earlier than Z. Hence, Z sets later than H; but it has been shown to rise earlier. Therefore, Z rises earlier than H but sets later.

Paraphrase. If certain stars rise and set, and if they are on the circumference of a great circle some of whose points neither rise nor set, then the more northerly of the stars both rise earlier and set later.

Remark. This proposition is like the previous one in that its principal applications are in Propositions 9, 14 and 15, where the 'great circle' in the hypothesis is the ecliptic. And, contrary to the situation in Proposition 4, to say that some points of the ecliptic neither rise nor set is the same as saying that the Sun does not always rise and set each day, that is, one is in the latitudes of the so-called midnight Sun. It is clear that, in applying this proposition to two stars, they must lie on the same semicircle of the given great circle between its extreme northerly and southerly points.

Proof. As in Proposition 4, AGB, ADE, QZK, and LNM represent (respectively) the horizon, the greatest ever-visible circle, the parallel circle of the star further north, and the parallel circle of the star further south. Let great circles EZN and DZX define points N and X on the same parallel as the more southerly star, H. The Asymptotic Semicircles Theorem is used to show that N and X rise and set (respectively) with Z. Euclid now lets the diagram embed the following argument into the proof:

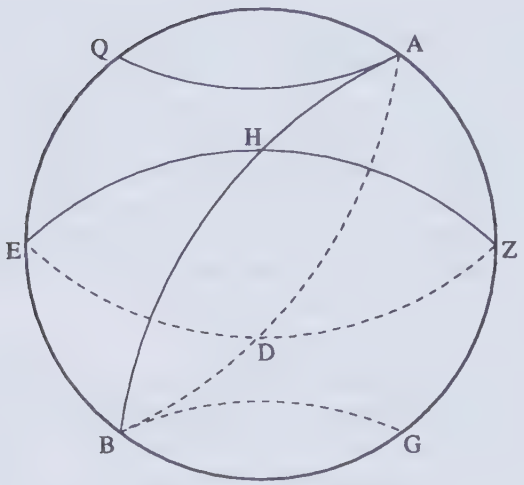


Figure 16. Proposition 6

The great circle GZB passes through one point of intersection of the great circles DZX and EZN; so, it passes through their other point of intersection. It therefore cuts any parallel circle south of QK, such as LHM, between its intersections with DZX and EZN. Thus, H lies to the east of N, so that N (and, hence, Z) rises before H. Similarly, H lies to the west of X, and so X (and, hence, Z) sets after H.

That some tacit argument has been introduced into the proof should not surprise us, since nothing in the proof seems to presuppose the hypothesis.

[Proposition] 6. *Diametrically opposite stars on the ecliptic rise and set in syzygy,*[43] *and similarly those on the equator.*

In the cosmos [Figure 16], let ABG be the horizon; let the ecliptic have
[32] position $AHBD$; let EZD be the equator; and let segments AHB [and] EHZ be above the earth.

Then, point A is diametrically opposite to B; and E, to Z.

I say that points A [and] B and points E [and] Z rise and set in syzygy.

Let the eastern side be AG, the western BE; let [the] parallel circles on which points A [and] B move be AQ [and] BG; and let segment AQ be above the earth and BG [be] below the earth.

[43] See 47n24, above.

Since point A is diametrically opposite to B and E to Z, arc EB is equal to arc AZ. But EB is equal to ZG; and, therefore, AZ is equal to ZG. And EZD is the largest of the parallels; therefore, circle AQ is equal to circle BG. And AQ [and] BG are alternate segments of them; therefore, arc AQ is equal to arc BG.

Therefore, when point A traverses arc AQ and comes to Q, point B traverses [arc] BG and comes to G. But A, when it has traversed AQ and reached Q, is setting; and B, when it has traversed BG and reached G, is rising. Therefore, as A is setting, B is rising.

And, similarly, we shall show that also, as A is rising, B is setting.

Again, since each of EHZ [and] ZDE is a semicircle, arc ZDE is equal to arc EHZ. Therefore, when Z traverses arc $Z[H]E$ and comes to E, point E traverses arc EDZ and comes to Z. But point Z, when it has [34] traversed $Z[H]E$ and reached E, is setting; and E, when it has traversed arc EDZ and reached Z, is rising. Therefore, as Z is setting, E is rising.

Then, similarly, we shall show also that, as Z is rising, E is setting.

Similarly, all diametrically opposite stars on the ecliptic and on the equator rise and set in syzygy.

Paraphrase. When one of two diametrically opposite stars on the ecliptic or equator is rising the other is setting, and *vice versa*.

Remark. The proposition is, of course, true for any two diametrically opposite stars that actually rise and set, simply because the horizon is a great circle of the sphere; so, if the horizon contains one of two diametrically opposite points, it must contain the other. That Euclid does not state such a general theorem is further evidence that this treatise is not a general introduction to rising and setting phenomena, but is specifically directed to risings of ecliptic and equatorial stars, that is, to the problem of rising times.

In any case, Euclid actually proves that, if one of two stars (both of which are either on the ecliptic or on the equator) is rising while the other is setting, then, when the former is setting, the latter is rising. In other words, he demonstrates that, if two stars are diametrically opposite, then they remain so when the sphere has rotated. He assumes the reader can supply the argument that, if two stars on a great circle are diametrically opposite, then, when one is rising, the other is setting.

Proof. With reference to the diagram, stars A and B on the ecliptic are taken on the horizon. Such stars are diametrically opposite, says Euclid; and, therefore, arc AZ is equal to arc EB. (This is elementary plane geometry, since the diameters AB and EZ intersect at the centre of circle ABG.) But, because BG is the parallel circle of star B, it follows from

the latter part of the Asymptotic Semicircles Theorem that arc EB equals arc ZG; and, therefore, that arc AZ equals arc ZG. By the Theorem of Parallels, the parallel circles AQ and BG are equidistant from the equator and so, given the introduction to the treatise, the part of AQ above the earth is equal to the part of BG below the earth. (The text's 'circle AQ is equal to circle BG' is correct but unnecessary. It is the equality of alternate segments, stated next, which is relevant. We suspect an interpolation.) Thus, in the time A traverses AQ the star B traverses the arc BG; so, when A sets at Q, the star B rises at G.

The rest of the proof for the ecliptic is obvious; and the proof for stars E and Z on the equator is immediate from the fact that the two equatorial arcs between them are semicircles and, thus, equal. (However, its truth for the equator is assumed in the introduction in the argument that the ecliptic and equator are great circles.)

There are problems with the diagram in Menge's text, for it makes it appear that A and B are the northernmost and southernmost points on the ecliptic and that, for example, AQ is the northern tropic; whereas this is not in fact assumed.

[Proposition] 7. *The ecliptic rises and sets at every place on the horizon between the tropics when the largest of the ever-visible [circles] is not larger than the tropic, and [the ecliptic] makes its turns*[44] *moving in opposite directions. For, when it moves southward in the east, it appears moving northward in the west; but, when it moves northward in the east, it appears moving southward in the west. And at different times it stands over us differently.*

In the cosmos [Figure 17], let ABG be the horizon; AD, the summer tropic; BG, the winter [tropic]; let the ecliptic have position DEB; and let segment DEB be below the earth, and DZB [be] above the earth.

I say that the ecliptic rises and sets at every place on the horizon between the tropics, and that it makes its turns moving in opposite directions. For, when it moves southward in the east, it is seen moving northward in the west; and, when it moves northward in the east, it is seen moving southward
[36] in the west. And at different times it stands over us differently.

Let the eastern side be DG and the western AB.

[*Part 1*]. Now, [it is] clear that the ecliptic rises and sets at every place on the horizon between the tropics, since it touches larger circles [the tropics] than those the horizon touches.

[44] For 'turns,' see 47n23 above.

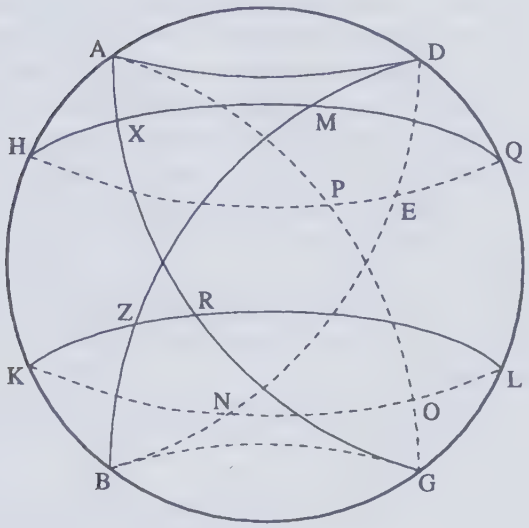


Figure 17. Proposition 7

[*Part 2*]. I say that it makes its turns moving in opposite directions.

[*Proof of Part 2.1*]. When the ecliptic moves southward in the east, it is seen to move northward in the west.] Let DE [and] ZB be chosen as equal and opposite arcs;[45] and let there be drawn parallel circles HEQ [and] KZL on which points E [and] Z move.

Since arc DE is equal to arc ZB, let there be added the common [arc] EB; then, the whole DEB is equal to the whole EBZ. And DEB is a semicircle; therefore, EBZ too is a semicircle. Therefore, point E is diametrically opposite point Z. And, since arc ED is equal to arc DM and ZB [is equal] to BN (but DE is equal to ZB), DM is also equal to BN. [38]

Let there be added the common [arc] MB; then, the whole DMB is equal to the whole MBN. But DMB is a semicircle; therefore, MBN too is a semicircle. Therefore, point M is diametrically opposite point N. Since diametrically opposite points of the ecliptic rise and set in syzygy, when point D is rising at point D, the diametrically opposite [point] B is setting at point B; and when [point] E is rising at point Q, the diametrically opposite [point] Z is setting at point K; and, when point N is rising at point L, the diametrically opposite [point] M is setting at point H; and, moreover, when point B is rising at point G, the diametrically opposite [point] D is setting at point A. Therefore, when the ecliptic moves southward in the east it appears moving northward in the west.

[45] For 'opposite' as applied to arcs, see English Glossary *s.v.*

[*Part 2.2*]. I say in addition that, when it moves northward in the east, it is seen to move southward in the west.

[*Proof*]. When semicircle DEB has risen, the ecliptic will have position AXG. And, in the same way, we shall show that point X is diametrically opposite to point O, and R [diametrically opposite] to P.

When point G rises at point G, the diametrically opposite [point] A sets at point A; and, when point O rises at point L, the diametrically opposite [point] X sets at point H; and when point P rises at point Q, the diametrically opposite point R sets at point K; and, moreover, when point A rises at point D, the diametrically opposite point G sets at point
[40] B. It follows that, when the ecliptic moves northward in the east, it is seen to move southward in the west.

But it has been shown also that, when it moves southward in the east, it is seen to move northward in the west.

[*Part 3*]. And [it is] clear that at different times it stands over us differently.

[*Proof*]. When the point of contact of the ecliptic and the summer tropic is at the midpoint of the segment of the tropic above the earth, it will be most upright to us; but, when it is at the midpoint of the segment of the tropic below the earth, it will be lowest to us. And the farther it gets from the midpoint of the segment of the summer tropic above the earth, the more inclined it will be; it will be similarly inclined when equally distant from either of the midpoints.

Paraphrase. If every point of the ecliptic rises and sets, then the ecliptic rises and sets at every place on the horizon between the tropics; and, as its intersection with the eastern horizon moves southward (northward) during the course of the daily rotation of the heavens, its intersection with the western horizon moves northward (southward). And at different times it stands over us differently.

Remark. The condition that every point on the ecliptic rises and sets each day, which we met also in Proposition 4 (at least tacitly, in terms of its application in this work), means that the locality is between what are at present called the arctic and antarctic circles. So, there is no midnight Sun. Interestingly enough, when it comes to justifying the first part of the conclusion, Euclid does no more than recast the hypothesis in the form that the ecliptic 'touches larger circles [that is, the tropics] than those the horizon touches,' which is precisely the wording of *De sphaera* Theorem 11, where the first part of the conclusion is proved.

Proof. To prove the second part, Euclid first shows that, when the point on the ecliptic that is rising is moving south along the eastern horizon, the

point that is setting is moving north along the western horizon, to which end he takes an invisible arc DE of the ecliptic, with D on the eastern horizon, and an equal, visible arc BZ with B on the western horizon. The result of adding the common (invisible) arc EB to these two equal arcs is that arc DEB is equal to arc EBZ. Since DEB is a semicircle, so is EBZ; and, thus, its endpoints, E and Z, are diametrically opposite. By Proposition 6, they therefore rise and set in conjunction; that is, when E rises at Q, the star Z is setting at K. He now lets the ecliptic also intersect the parallel circles HEQ at M and LZK at N, and he easily shows that M and N are diametrically opposite points, so that, when N is rising at L, the star M is setting at H. Finally, because B and D are diametrically opposite, when B is rising at G, the star D is setting at A. Thus, as the rising point moves southward along the eastern horizon from D to Q to L, the setting point moves northward along the western from B to K to H.

The proof that, as the rising point moves northward along the eastern horizon, the setting point moves southward along the western, is entirely similar; and Euclid omits the proofs, here as in Proposition 6, that the relevant points are diametrically opposite.

As for the proof of the third part, that at different times the ecliptic stands over us differently, Euclid draws our attention to the fact that, when the beginning of Cancer is on the visible part of the meridian, the ecliptic will be most upright; and that, when it is on the invisible part of the meridian, the ecliptic will be least upright—both of which facts follow from the Inclined Arcs Theorem.

Note that, in the proof of the second part, Proposition 6 is used as it is stated, not as it is proved. Note also that, in the proof of the same part, the justification for DEB's being a semicircle is the fact stated in the introduction that the horizon bisects the 'circle of the signs'. Moreover, the argument that E and Z are diametrically opposite establishes the following geometrical lemma:

> If DEB and BZD are semicircles of a single great circle, disjoint except for their endpoints, and if arc DE is equal to arc BZ, then E and Z are diametrically opposite.

[Proposition] 8. *The signs rise and set in unequal segments of the horizon; those at the equator in the largest, those next them in smaller [segments], and those at the tropics in the smallest. The [signs] equidistant from the equator [rise and set] in equal [segments of the horizon].*

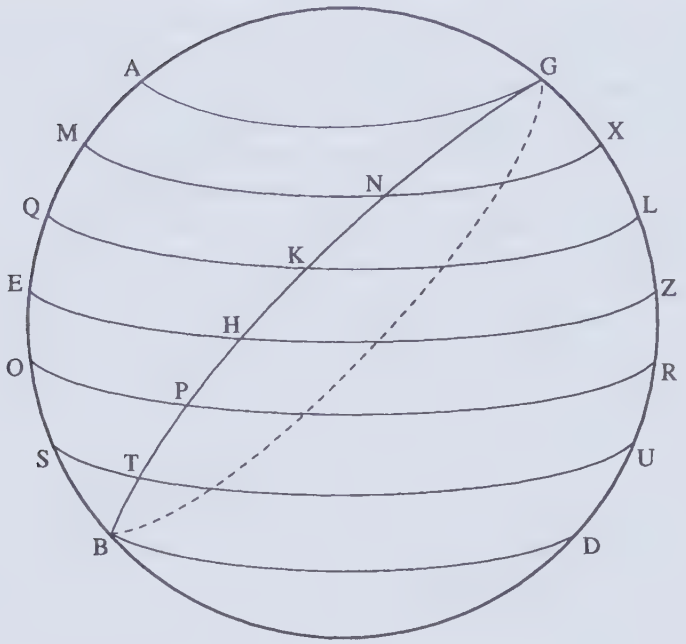


Figure 18. Proposition 8

In the cosmos [Figure 18], let $ABGD$ be the horizon; AG [and] BD, the tropics; let the ecliptic have position BG; [let] EZ [be] the equator; and let each of GH [and] HB be trisected at the points N [and] K, P [and] T.

I say that the arcs GN, NK, KH, HP, PT, [and] TB rise and set in unequal segments of the horizon; [that] KH [and] HP [set] in the largest; KN
[42] [and] PT in smaller; GN [and] BT in the smallest [segments]; and that KH and HP, KN and PT, [and] NG and TB [rise and set] in equal [segments].

Let MX, QL, OR, [and] SU be the parallel circles on which points N, K, P, [and] T move.

Since the arcs HK, KN, [and] NG are equal to one another, [arcs] ZL, LX, [and] XG are successively shorter beginning from the longest, ZL. And also, for the same [reasons, arcs] EQ, QM, [and] MA are successively shorter beginning from the longest, EQ; and moreover [arcs] ZR, RU, [and] UD are successively shorter beginning from the longest, ZR; moreover, [arcs] EO, OS, [and] SB are successively shorter beginning from the longest, EO. Since GN, NK, KH, HP, PT, [and] TB rise throughout arcs GX, XL, LZ, ZR, RU, [and] UD and set throughout AM, MQ, QE, EO, OS, [and] SB, they rise and set in unequal segments of the horizon. And, since the parallel circles QL [and] OR in the sphere cut off equal arcs

PH [and] HK from a great circle GB at the largest of the parallels EZ, circle QL is equal to circle OR. Now, since the equal and parallel circles QL [and] OR in the sphere cut off equal arcs LZ [and] ZR from the great circle $ABGD$ at the largest of the parallels EZ, arc LZ is equal to arc ZR.

Similarly, then, we shall show that arc XZ is equal to arc ZU. Therefore, [44a] the remainder, XL, is equal to the remainder, RU. Then, also for the same [reasons] GX is equal to UD.

Therefore, the signs rise and set in unequal segments of the horizon; those at the equator in the largest, [then] successively in smaller [segments], and those at the tropics in the smallest [segments]. The [signs] equidistant from the equator [rise and set] in equal [segments of the horizon].

Paraphrase. The zodiacal signs rise and set in unequal segments of the horizon; those at the equator in the largest segments, those next to them in smaller, and those at the tropics in the smallest. Signs equidistant from the equator rise and set in equal segments.

Remark. This proposition makes it clear that the phrase, 'the zodiacal signs,' refers only to the three signs in any one of the four quadrants of the zodiac defined by the equinoctial and the solstitial points. It is interesting that here Euclid restricts the last part of Proposition 8 to zodiacal signs, whereas in Proposition 12 he makes a broader claim than he justifies in the proof.

Proof. Since the enunciation presumably speaks of all the signs of the ecliptic rising and setting, it tacitly assumes, as in Proposition 7, that the locality is between the (modern) arctic and antarctic circles. The segments of the horizon where the signs rise and set are those determined by the parallel circles through the ends of the signs, and the inequality of these segments is an immediate consequence of the Horizon Arcs Theorem, with E taken to be the ecliptic and H the horizon, as is the decreasing size of the segments as one moves away from the equator. The equality of segments where signs equidistant from the equator rise follows from the Theorem of Parallels. (One sees how a reader uninstructed in spherics might be completely lost in this work.)

[a] [Proposition] 9. *Whenever the north pole*[46] *is over the horizon and, moreover, the largest of the ever-visible [circles] is smaller than the summer*

[46] Everywhere else Euclid presupposes the visibility of the celestial north pole, simply called the 'visible pole' in the introduction. The definition of largest of the ever-visible circles in the introduction depends on the visibility of the celestial

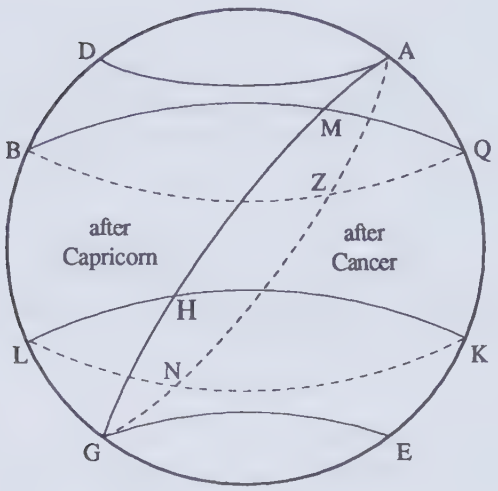


Figure 19. Proposition 9a

tropic, semicircles of the ecliptic that do not begin on the same parallel rise in unequal times, [with] the [semicircle] following Cancer [rising] in the greatest [time], those [semicircles] following, each in turn, in lesser [times], and the [semicircle] following Capricorn in the least. And those [semicircles] that begin on the same parallel rise in equal times.

[46a] In the cosmos [Figure 19], let ABG be the horizon; DA, the summer tropic; EG, the winter tropic; let the ecliptic have position AGZ; let AZG be the semicircle following Cancer below the earth, and GHA be the semicircle following Capricorn above the earth. Now, let equal arcs GH [and] ZA be chosen such that Z is diametrically opposite H; and let the parallel circles on which points Z [and] H move be BZQ [and] KHL. Then, M is diametrically opposite N.

[48a] [*Part 1*]. Now, I say that semicircle AZG rises in the greatest time, and next in order ZGH, and after this NGM, and in the least [time] GMA.

Proof. Since arc DA is longer than [an arc] similar to BMQ,[47] and BMQ [is longer] than [an arc similar] to LHK, and LHK [is longer] than [an arc similar] to GE, A traverses arc AD in a greater time than Z (beginning

north pole. This is also the only place where the specification 'north pole' is used. Likewise, that the largest of the ever-visible circles is smaller than the tropics is normally presupposed [cf. p. 32, above].

[47] Such phrases as μείζων/ἐλάττων ἢ ὁμοία (longer/shorter than similar to) are technical expressions used several times in Euclid's text, their meaning being as expanded in our translation.

from Q) traverses QMB, and Z (beginning from Q) traverses arc QMB in a greater time than N (beginning from K) traverses KHL, and N (beginning from K) traverses arc KHL in a greater time than G (beginning from E) traverses EG. But, while A (beginning from A) traverses arc AD, the diametrically opposite [point] G (beginning from G) traverses the [arc] [50a] below the earth of the circle GE, and semicircle AZG rises. And, while Z (beginning from Q) traverses arc QMB, the diametrically opposite [point] H too (beginning from L) traverses LNK, and semicircle ZGH rises. And, while N (beginning from K) traverses arc KHL, the diametrically opposite [point] M too (beginning from B) traverses arc BZQ, and semicircle NGM rises. And, while G (beginning from E) traverses arc EG above the earth, the diametrically opposite point A (beginning from D) traverses arc DA below the earth, and semicircle GMA rises. Therefore, AZG, that is, the [semicircle] following Cancer, rises in the greatest time, and next ZGH, after this NGM, and GMA, that is, the [semicircle] following Capricorn, in the least time.

[*Part 2*]. And I say also that those [semicircles] that begin on the same [52a] parallel circle rise in an equal time.

[*Proof*]. In the time in which Z (beginning from Q) traverses arc QMB, M (beginning from Q) traverses arc QMB; but, while Z (beginning from Q) traverses QMB, the diametrically opposite [point] H (beginning from L) traverses arc LNK, and semicircle ZGH rises. And, while M (beginning from Q) traverses QMB, the diametrically opposite [point] N (beginning from L) traverses LNK, and semicircle MAN rises. It follows that semicircles ZGH [and] MAN rise in an equal time. Therefore, semicircles of [54a] the ecliptic that begin on the same parallel rise in an equal time.

[**b**] [Proposition] 9. *Semicircles of the ecliptic that do not begin on the* [44b] *same parallel rise entirely in unequal times, [with] that following Cancer in the greatest, those following, each in turn, in lesser [times], and that following Capricorn in the least. And those [semicircles] beginning on the same parallel rise in equal times.*

In the cosmos [Figure 20], let ABG be the horizon; AD, the summer tropic; [46b] BG, the winter tropic; and let the ecliptic have position $DEBZ$. Let GD be the eastern side, AB, the western; let DEB be the semicircle following Cancer, and BZD the [semicircle] following Capricorn.

[*Part 1*]. I say that the semicircles of the ecliptic that do not begin on the same parallel rise in unequal times, DEB following Cancer in the greatest, those following, each in turn, in lesser [times], and BZD following [48b] Capricorn in the least. ([Part 2] And those [semicircles] that begin on the same parallel rise in equal times.)

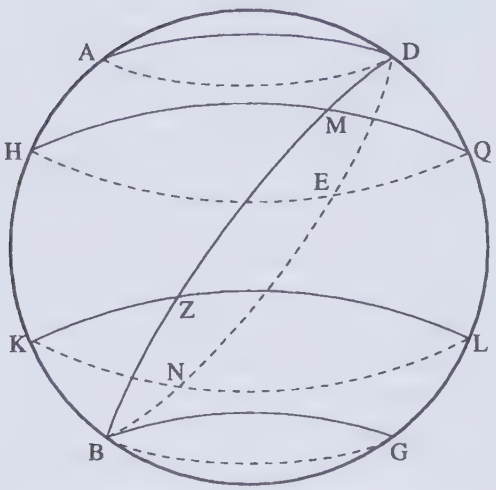


Figure 20. Proposition 9b

Let equal arcs DE [and] BZ be chosen and let there be drawn the parallel circles $HEQM$ [and] $KZLN$ on which the points E [and] Z move. And let the segments of them above the earth be HMQ [and] KZL. Now, we shall show as before that point E is diametrically opposite to point Z and M [is diametrically opposite] to N.

Since [the visible] arc AD is longer than [an arc] similar to arc HMQ, and HMQ [is longer than an arc similar] to KZL and, moreover, KZL [is longer than an arc similar] to BG, point D (beginning from D) traverses arc AD in a greater time than E (beginning from Q) traverses arc QMH; and E (beginning from Q) traverses [arc] QMH in a greater time than N (beginning from L) traverses arc LZK; and N (beginning from L) traverses [arc] LZK in a greater time than B (beginning from G) traverses [the
[50b] visible] arc GB. But, while point D traverses arc DA, the diametrically opposite point B traverses the [invisible] alternate arc BG, and semicircle DEB rises. And, while E (beginning from Q) traverses arc QMH, the diametrically opposite [point] Z (beginning from K) traverses KNL, and semicircle EBZ rises. And, while N (beginning from L) traverses LZK, the diametrically opposite [point] M (beginning from H) traverses HEQ, and semicircle NBM rises. And, while B (beginning from G) traverses GB, the diametrically opposite [point] D (beginning from A) traverses the
[52b] alternate [arc] AD, and semicircle BZD rises.

Therefore, semicircle DEB following Cancer rises in the greatest time, and EBZ in a lesser [time] than DEB, and NBM in a still lesser [time] than EBZ, and BZD following Capricorn in the least [time].

[*Part 2*]. I say that those [semicircles] that begin on the same parallel rise in equal times.

[*Proof*]. Let semicircles MDN [and] EBZ begin on the same parallel.

I say that semicircles MDN [and] EBZ rise in equal times.

The point M (beginning from Q) traverses arc QMH, and E (beginning from Q) traverses arc QMH in an equal time. But, while point M (beginning from Q) traverses QMH, the diametrically opposite [point] N (beginning from K) traverses arc KNL, and semicircle MDN rises. And, while point E (beginning from Q) traverses arc QMH, the diametrically [54b] opposite [point] Z (beginning from K) traverses KNL, and semicircle EBZ rises. It follows that semicircles MDN [and] EBZ rise in an equal time.

Paraphrase. For localities north of the equator and south of the (modern) arctic circle, the semicircles of the ecliptic that do not begin on the same parallel rise in unequal times, that following Cancer in the greatest, those following, each in turn, in lesser times, and that following Capricorn in the least. And semicircles beginning on the same parallel rise in equal times.

Remarks. (1) If the north pole is not over the horizon it is (since we are dealing with northern latitudes) on the horizon, which means that the locality is on the equator. Here all days have equal length, so all semicircles of the ecliptic rise in equal times. On the other hand, north of the arctic circle, some summer days are twenty-four hours long; so some parts of the ecliptic do not rise at all.

(2) The first sentence of the proposition clearly concerns semicircles whose initial point falls somewhere in the signs in the semicircle following Cancer, that is, Cancer, Leo, Virgo, Libra, Scorpio, and Sagittarius.

Proof. In the proof of the first part, Euclid assumes that the semicircle following Cancer is below the earth, and he begins just as in Proposition 4—a fact he indicates with the words 'we shall [meaning 'could'] show as before' in **b**—by taking equal arcs of the ecliptic, AZ below the earth and GH above the earth. The parallel circles through Z and H give him two other points on the ecliptic, M and N respectively, defined by the other intersections of the ecliptic with the parallel circles, and said to be diametrically opposite. And he says he will now show that the rising times of AZG, ZGH, NGM, and GMA progressively decrease.

Now, because arcs AD, QMB, KHL, and EG are the visible portions of increasingly southerly parallel circles, each is longer than similar to the following, by the remarks in the introduction. Hence, A, Z, N, and G traverse these visible arcs in decreasing times. Hence, the points diametrically opposite them (namely, G, H, M, and A) traverse their invisible

arcs in decreasing times. But, as they do that, semicircles AZG, ZGH, NGM, and GMA rise; so, they rise in progressively decreasing times, as was claimed.

Further, Z and M traverse the visible parts of their parallels in equal times, and as each does that the point diametrically opposite it (H and N, respectively) traverses the invisible part of its parallel. But this means that in those times semicircles ZH and MN rise. This proves the last part of the proposition.

Other than **b**'s reference to a similar point in the proof of Proposition 4, where Euclid asserts that M is opposite N, and **b**'s use of different letters to refer to points in the diagram, there is no significant difference between the proofs in recensions **b** and **a**. Note, however, that **b** omits the essential conditions stated at the beginning of **a**'s version of Proposition 9.

[Proposition] 10. *If two semicircles of the ecliptic having an arc in common rise in unequal times, then the opposite arcs also rise in unequal times, and there will be the same difference in the times in which the semicircles rise and [in which] the opposite arcs rise. And, if two semicircles of the ecliptic having an arc in common rise in an equal time, then the opposite arcs also rise in an equal time.*

[**a**: Figure 21] In the cosmos, let ABG be the horizon, let the ecliptic have position $AEGD$, and let AD [and] GE be chosen to be equal arcs (therefore, D is diametrically opposite E), and let semicircles ADG [and] DGE rise in unequal times.

I say that the opposite arcs AD [and] GE rise in unequal times too, and
[56a] there is the same difference in the times in which semicircles ADG [and] DGE rise and in which arcs AD [and] GE rise.

[*Proof*]. Since the semicircles ADG [and] DGE rise in unequal times, let the rising time of DG be subtracted (for, the arc DG always rises in a time equal to itself); then, the remaining arcs AD [and] GE rise in unequal times, and the differences are the same between the rising times in which semicircles ADG [and] DGE and the opposite arcs AD [and] GE rise.

Then, again, [suppose that] the semicircles ADG [and] DGE rise in equal times. Let the common [rising] time of arc GD be subtracted; then, the remaining [arcs] AD [and] GE rise in an equal time.

[**b**: Figure 22] Let the circle $ABDG$ be the horizon; AG, the summer tropic; BD, the winter [tropic]; GB, the ecliptic; and let GE [and] BZ be chosen to
[56b] be equal arcs. Therefore, semicircles GEB [and] EBZ rise in unequal times.

I say that arcs GE [and] BZ rise in unequal times.

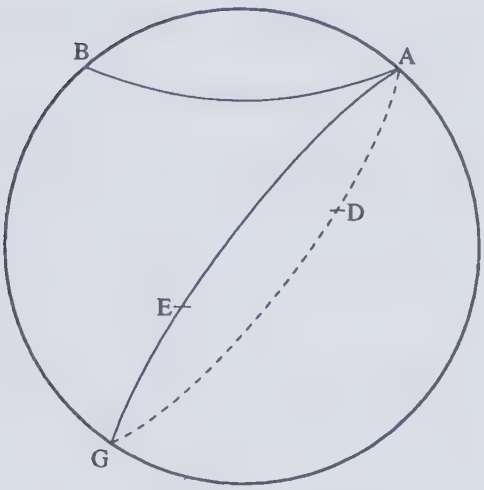


Figure 21. Proposition 10a

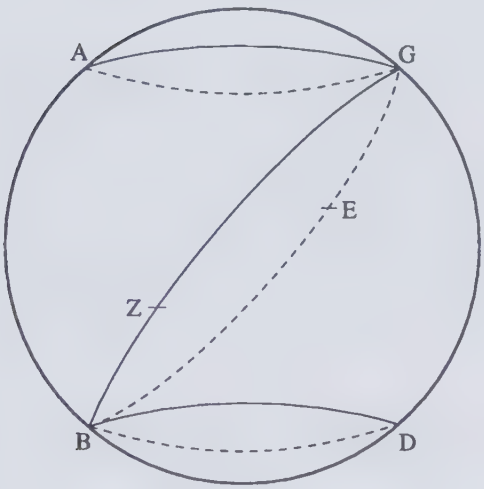


Figure 22. Proposition 10b

[*Proof*]. Since GEB rises in a greater time than EBZ, let the time of rising of arc EB be taken away (for the arc EB always rises in a time equal to itself). Then, the remaining arc GE rises in a greater time than BZ, and [it is] clear that the differences are the same between the times in which semicircles GEB [and] EBZ and opposite arcs GE [and] BZ rise.

And it is clear that, if some semicircles rise in equal times, the opposite arcs also rise in equal times.

Paraphrase. If two semicircles of the ecliptic having an arc in common rise in unequal times, then the opposite arcs also rise in unequal times; and there is the same difference in the times in which the semicircles rise and the opposite arcs rise. And, if two semicircles of the ecliptic circle having an arc in common rise in equal times, then the opposite arcs also rise in equal times.

Remark. This proposition serves as a lemma allowing one to deduce equalities or inequalities about rising times of individual signs (or even smaller arcs of the ecliptic) from the equalities or inequalities about rising times of semicircles proved in Proposition 9. The proof, of course, applies equally well to arbitrary ecliptic arcs having an arc in common and, had it (or at least the statement in the second sentence of the proposition) been proved in this generality, it would have been useful in the proof of Proposition 12, where it is tacitly assumed.

Proof. The rising times of semicircles ADG and DGE are the sums of the rising times of AD and DG and of DG and GE respectively. If these sums are unequal, then, when we subtract the rising time of the common arc DG, the rising times of the arcs AD and GE are unequal. But these arcs are also opposite; so, the first part of the proposition is proved, and the same idea suffices to prove the second.

The idea of the proof in **b** is the same as in **a**, but the author has made the unnecessary assumption that one of the semicircles follows Cancer.

[58a] [Proposition] 11. *Of [two] equal and opposite arcs of the ecliptic, while the one rises the other sets; and, while the one sets, the other rises.*

[**a**: Figure 23] In the cosmos, let ABG be the horizon, let the ecliptic have position $AEGD$, let ADG be the semicircle below the earth, and let AD [and] GE be chosen as equal and opposite arcs.

I say that, while arc AD rises, arc GE sets; and [that,] while [arc] GE rises, arc AD sets.

[*Proof*].[48] Let the parallel circles on which the points E [and] D move be EQB [and] KDL. Since diametrically opposite stars on the ecliptic rise

[48] There are serious problems with the following proof, the least of which is that, as Menge [1916, 59n] observed, the letters H and Z play no role in the argument presented. Rather than change the text or diagram arbitrarily in order

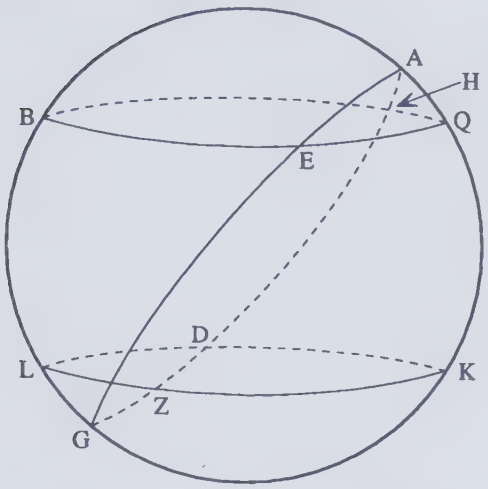


Figure 23. Proposition 11a

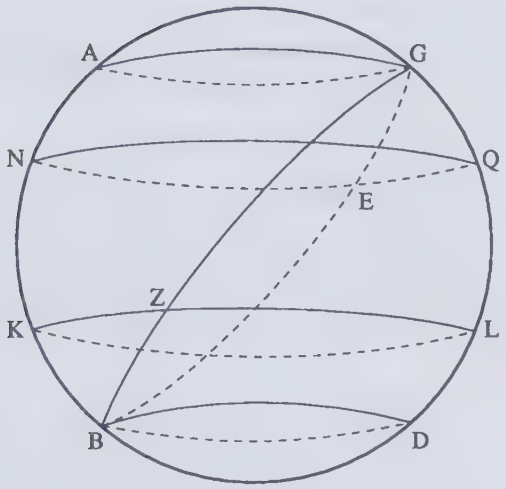


Figure 24. Proposition 11b. Part 1

and set in syzygy, when E rises at Q, the diametrically opposite point D sets at L. Therefore, while E traverses arc EQ and comes to Q, D traverses DL and comes to L. But, while D traverses DL, DA rises, and, while E traverses EQ, GE sets. Therefore, while DA rises, GE sets. [60a] [62a]

to make them agree, we present both as they are Menge's text. Any reader can put together a correct proof.

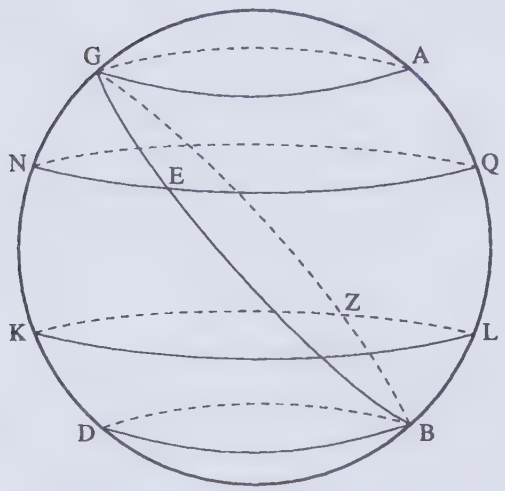


Figure 25. Proposition 11**b**. Part 2

Similarly, we shall show that, while AD sets, GE rises.

[58b] [**b**] Let the circle $ABDG$ be the horizon; AG, the summer tropic; BD, the winter [tropic]; GB be the ecliptic; and let GE [and] BZ be chosen as equal and opposite arcs on [the ecliptic].

[*Part 1.* Figure 24]. I say that, while GE rises, BZ sets.

Let NQ [and] KL be parallel circles on which points E [and] Z move. Since diametrically opposite stars on the ecliptic rise and set in syzygy, when E rises, Z sets. Therefore, while E (beginning from E) traverses arc
[60b] EQ and comes to Q, Z (beginning from Z) traverses ZK and comes to K. But, when E traverses EQ and comes to Q, arc EG rises; and, when Z traverses ZK and comes to K, arc BZ sets. Therefore, while arc GE rises, arc BZ sets.

[*Part 2.* Figure 25]. I say also that, while BZ rises, GE sets.

[*Proof*]. In the second case, let the ecliptic be rotated and have position $GEBZ$. Since point Z is diametrically opposite point E, E sets when Z rises. Therefore, while Z traverses arc ZL and comes to L, E traverses arc
[62b] EN and comes to N. But, when Z traverses arc ZL and comes to L, BZ rises; and when E traverses arc EN and comes to N, GE sets. Therefore, while arc BZ rises, arc GE sets.

Paraphrase. Of equal and opposite arcs of the ecliptic, the one sets in the time in which the other rises, and rises in the time in which the other sets.

Remark. The proof for the first part in **a** seems to have become joined to the diagram for the second part. In any case, we follow the proof in **b**, which is the same as that apparently intended in **a**. Proposition 12 also requires two diagrams, one for setting and one for rising. That **b** has two diagrams and two arguments for both Propositions 11 and 12, whereas **a** has only one diagram and one incomplete proof for each, is further evidence for the superiority of **b** at this point. Despite this, the proof in **a**, for all its shortcomings, deals with the general case; whereas the proof in **b** assumes, unnecessarily, that the equal arcs begin at the solstitial points.

Proof [Figure 24]. Euclid takes two equal arcs of the ecliptic, GE in the invisible hemisphere with its endpoint G on the horizon, and the other BZ in the visible hemisphere with its endpoint B on the horizon. Euclid now considers the parallel circles through the other endpoints, namely, QEN and LZK. Since E and Z are diametrically opposite, it follows from Proposition 6 that, when E is rising, Z is setting; but, when E is rising, arc GE has risen and, when Z is setting, arc BZ has set. Thus, GE rises in the same time in which BZ sets. The proof for the second half involves no new ideas, only another diagram.

This proposition is like the previous one in that it is used in the proofs of Proposition 13 and the lemma following Proposition 13.

[Proposition] 12. *Equal arcs*[49] *of the semicircle following Cancer set in unequal times, those at the points of contact with the tropics in the greatest [time], those following them in lesser [times], and those at the equator in the least [time], but those equidistant from the equator both set and rise in an equal [time].*[50]

[**a**: Figure 26] In the cosmos, let ABG be the horizon; RST, the largest of the ever-visible [circles]; AE, the summer tropic; GZ, the winter tropic; BHD, the equator; and let the ecliptic have position AHG. Let AHG be the semicircle following Cancer above the earth; and let each of the quadrants AH [and] HG be divided into the signs at Q, K, L, [and] M. [64a]

I say that [Part 1] AQ, QK, KH, HL, LM, [and] MG set in unequal times; AQ [and] GM in the greatest [times]; QK [and] LM in lesser [times]; and KH [and] HL in the least [time]; but [Part 2] that in an equal time [there set] AQ and MG, QK and ML, KH and LH.

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[49] Actually, as the proof indicates, the standard twelfths of the ecliptic usually designated as signs.

[50] So **a**: **b** has 'in equal [times]'

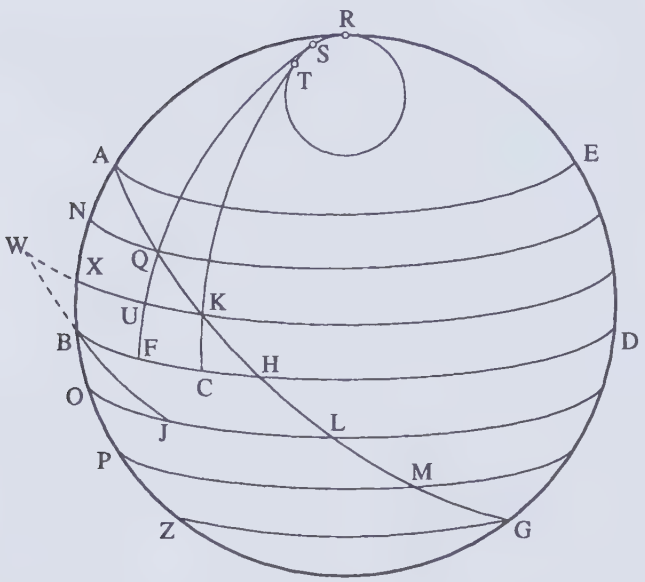

Figure 26. Proposition 12a

[66a] [*Beginning of Part 1*].

[*Proof*]. Let parallel circles NQ, XK, OL, [and] PM be drawn through Q, K, L, [and] M; and let great circles SQF [and] TKC be drawn through Q [and] K touching RS such that the [arcs] of the parallel circles between AR, SQF, [and] TKC are similar. That is, let them be drawn so that the semicircle from R in the direction of AN is asymptotic to the semicircles through S [and] T in the directions of C [and] F [cf. Tangent Circle Theorem].

[Arcs] NQ, XU, [and] BF correspond to [one another]; and, therefore, UK [and] FC [correspond to each other]. Hence, while Q traverses QN, U traverses UX, and F traverses FB. And, since arc QA sets while Q traverses QN, [arc] QA sets while U traverses arc UX and F [traverses] BF. Again, since [arc] KA sets while K traverses KX and C [traverses] CB, KQ sets [in the time] in which K traverses KU, that is, [while] C
[68a] [traverses] CF (for they are similar).

Then, for the same [reasons] also, [arc] HK sets [in the time] in which H traverses HC.

[1] Now the great circle ABG in the sphere touches a circle RST in the sphere.

[2] Another great circle AHG, which is oblique to the parallels, touches larger [parallel circles] than those [that] ABG touches.

[3] Also, [on AHG,] equal arcs AQ, QK, [and] KH have been cut off successively in the direction of the largest of the parallels BHD.

[4] Great circles SF [and] TC have been drawn through the points Q [and] K to touch the circle RST (which ABG also touched from the beginning),[51] and to make the semicircles from the points of contact S [and] T in the directions of K [and] Q be asymptotic to the semicircle RAB of the horizon on which lies the oblique circle's point of contact [with the summer tropic] (which is between the visible pole and the largest of the parallels). [70a]

Therefore, they [RAB, SQF, and TKC] will cut off between them unequal arcs of the largest of the parallel circles [BHD]. It follows that arc BF is longer than FC, and FC [is longer] than CH. Therefore, F traverses FB in a greater time than C traverses FC, and C [traverses] FC in a greater time than H [traverses] HC. But QA sets while F traverses FB, and QK sets [in the time] in which C traverses CF, and HK sets [in the time] in which H traverses HC. Therefore, AQ sets in a greater time than KQ, and QK [sets in a greater time] than KH.[52]

[*Part 2*]. Now, I say that also [arcs] equidistant from the equator set in an equal time.

[*Proof*]. Let the ecliptic be rotated, and let H come to B. Then, the ecliptic will have position WBJ. And KH is equal to HL; therefore, WB is equal to BJ. [72a]

Also BHD is the largest of the parallels, and circles WK [and] JL [are equal] parallels; therefore, WK is equal to JL. Hence, in addition, BX is also equal to BO; and, therefore, the [line segment] from W to X is equal to the [line segment] from O to J. Hence, arc WX is also equal to arc OJ. [74a] Therefore, [in the time] in which W (beginning from X) traverses XW, J also traverses JO. But BW sets while W (beginning from X) traverses XW, and BJ sets while J traverses OJ. Therefore, arcs WB [and] BJ set in an equal time. Hence, KH [and] HL also set in an equal time, and similarly QK [and] LM. Then, for the same [reasons], AQ [and] MG also set in an equal time.

[*Conclusion of Proof of Part 1*]. But AQ sets in a greater time than QK, and QK [in a greater time] than KH; therefore, GM too sets in a greater time than ML, and ML [in a greater time] than LH. [76a]

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[51] cf. [1], above.

[52] This proof is concluded after Part 2.

The same [things] will be shown also whenever the pole of the parallels is on the horizon and great circles [are] drawn through Q [and] K and through the pole <as they are drawn in the sixth theorem of the third book of the *Sphaerica* [of Theodosius]>.[53]

[78a] Therefore, equal arcs of the semicircle following Cancer set in unequal times, those at the points of contact with the tropics in the greatest [time], those following them in lesser [times], and those at the equator in the least [times], but those equidistant from the equator in equal [times].

[*Part 3*]. Then, similarly, we shall show that they also rise in an equal time to one another.

Paraphrase. Equal arcs of the semicircle following Cancer set in unequal time intervals, those at the points of contact with the tropics in the greatest, then those following them in less, and those at the equator in the least, but those equidistant from the equator both set and rise in equal times.

Remarks. (1) The last parts of this proposition and the following proposition on the equalities of rising times of signs equidistant from the equator, are proved in Ptolemy's *Almagest* ii 7, where they form one of his two key propositions leading to the computation of rising times. But Ptolemy's proof is quite different from Euclid's.

(2) Proposition 9 compares the rising times of all semicircles of the ecliptic; whereas Proposition 12 compares the rising times of individual signs, although only signs in the semicircle following Cancer.

(3) Although the statement of Proposition 12 speaks of arbitrary arcs following Cancer, the 'setting-out' in **a** and in **b** mentions only the six signs [see p. 5 Figure 1, above]. Thus restricted the proposition is true; whereas it would not be true of, say, two equal 10° arcs, the one centred on the beginning of Libra and the other being the last 10° of Libra.

The following account of the proof is specific to that in **a**. The account of the proof found in **b** follows the translation of that proof.

Euclid applies the Asymptotic Semicircles Theorem to the arcs cut off on the parallel circles through Q and K, the beginnings of the signs Leo and Virgo (respectively), by the disjoint semicircles starting at RAB (the western horizon), SQF and TKC (which may be regarded as the western horizon rotated eastward); and he concludes that the arcs cut off the parallel circles by a fixed pair of these semicircles are similar. In particular,

[53] The shift of geographical position indicated in the phrase 'when the pole of the parallels is on the horizon' is unusual in the *Phaenomena*. We suspect that the whole sentence is an interpolation.

arcs QN, UX, and FB are similar ('correspond' as **a** has it); so, while Q traverses QN (or, equivalently, while AQ, Cancer, sets) U traverses UX and F traverses FB. But the similarity of arcs KU and UX to arcs CF and FB, respectively, implies that, while AQ sets, K traverses KU and C traverses CF. Again, the similarity of arcs KX and CB implies that, while K traverses KX (that is, while KA sets), C traverses CB. It follows by subtraction that in the time that K traverses KU arc KQ (that is, Leo) sets. Similarly, in the time that H traverses HC arc HK (or Virgo) sets. The Equator Arcs Theorem now shows that arc BF is larger than arc FC, which is itself larger than CH. It is immediately apparent that AQ sets in greater time than does QK, which sets in greater time than KH. So, Part 1 of the proposition is proved for quadrant AH.

To prove Part 2 let the ecliptic rotate until KHL comes to WBJ. Since KH equals HL, their rotated images, WB and BJ, are also equal. Moreover, since circles WK and JL are parallel, it follows from the Theorem of Parallels that they are equal and, hence, by the same theorem, that XB equals BO. It follows from *Sphaerica* iii 3 that the line segments XW and OJ are equal; and, since they are chords of equal circles, arc XW equals arc OJ. But, then, in the rotation of the ecliptic, the points W and J would traverse arcs XW and JO, respectively, in equal times. But, while W traverses XW, arc BW sets; and, while J traverses JO, arc BJ sets. Hence, KH and LH set in equal times. Similarly, for arcs QK, LM and arcs AQ, MG.

This proves Part 2 of the proposition and, as an immediate consequence, Part 1 for the arcs of the ecliptic in the quadrant HG. The reason is that, by Part 2, the setting times for these arcs are equal to those of the arcs symmetrically located (relative to the equator) in the quadrant HA; thus, the inequalities for this latter quadrant can be applied to the arcs in quadrant HG.

[b] Let $ABGD$ be the horizon; EZ, the largest of the ever-visible [circles]; [62b] BA, the summer tropic; GD, the winter tropic; let BD be the semicircle following Cancer above the earth; HQ, the equator; and let each of BX [and] DX be divided into three equal parts by points K, L, M, [and] N.

I say that BK, KL, LX, XM, MN, [and] ND set in unequal times; [64b] BK [and] ND in the greatest [times]; KL [and] MN in lesser; and LX [and] XM in the least; but [that] in equal [times there set] LX and XM, KL and MN, BK and ND.

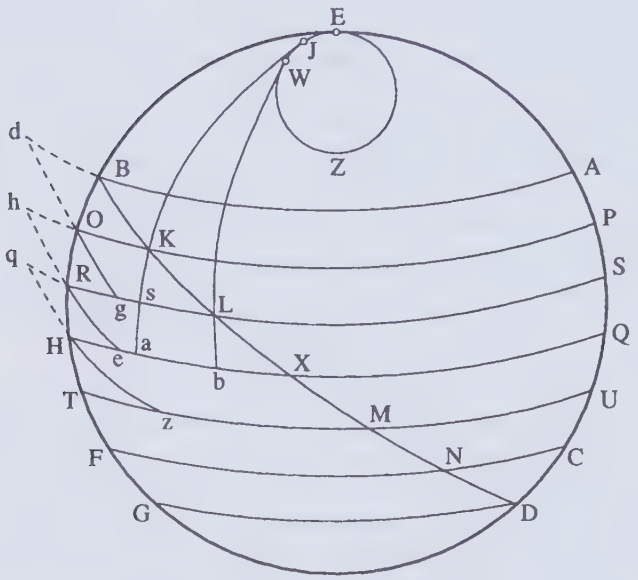

Figure 27. Proposition 12b. Part 1

[66b] [*Part 1.* Figure 27]. Let the parallel circles on which K, L, M, [and] N move be OP, RS, TU, [and] FC; and let great circles Ja [and] Wb be drawn through K [and] L touching the circle EZ.

Since BK, KL, [and] LX are equal to one another, Ha is longer than ab, [and ab is longer than] bX.[54]

Now, since Ha is longer than ab, but Ha is similar to OK, and ab [is similar] to sL, OK is also longer than [an arc] similar to sL and OK is shorter than [an arc] similar to LR.

Let Lg be similar to OK; then, while point K (beginning from K) traverses arc KO and comes to O, L (beginning from L) traverses Lg and comes to g, and the ecliptic will have position gOd.

[68b] Now, since arc OK is similar to gL, but OK is similar to Rs, Rs is also similar to gL. And they are [parts] of the same circle. Therefore, Rs is equal to gL.

Let the common [arc] gs be taken away; then, the remaining [arc] Rg is equal to sL. But OK is longer than [an arc] similar to sL; and, therefore, OK is longer than [an arc] similar to Rg.

[54] The Greek text has 'Ha, ab, bX are longer than one another, beginning from the longest Ha' [cf. Menge 1916, 66b.4–6].

Therefore, K traverses arc KO and comes to O in a greater time than g (beginning from g) [takes to] traverse arc gR and reach R. But arc BK sets while K traverses arc KO and comes to O; and arc KL sets in the time in which g traverses arc gR and comes to R. Therefore, BK sets in a greater time than KL.

Again, since ab is longer than bX, but ab is similar to sL, sL is longer than [an arc] similar to bX. Therefore, RL is much longer than [an arc] similar to bX but is shorter than [an arc] similar to HX.

Let Xe be similar to RL. Then, while X traverses arc Xe and comes to e, L traverses arc LR and reaches R, and the ecliptic will have position eRh.

Now, since RL is similar to eX, but RL is similar to Hb, Hb is similar [70b] to eX; and they are [arcs] of the same circle. Therefore, Hb is equal to arc eX. Let the common [arc] eb be taken away; then the remaining [arc] He is equal to bX. Since sL is longer than [an arc] similar to bX, but sL is equal to Rg (and bX [is equal] to He), Rg is longer than [an arc] similar to He.

Therefore, g traverses arc gR and comes to R in a greater time than e traverses eH and comes to H. But arc gO (that is, arc KL) sets while g traverses arc gR and comes to R. And eR (that is, arc LX) sets while e traverses eH and comes to H. Therefore, KL sets in a greater time than LX.

Again, since TM is longer than [an arc] similar to HX, let Mz be similar to HX. Then, while X (beginning from X) traverses arc XH and comes to H, M traverses arc Mz and comes to z, and the ecliptic will have position zHq.

Since the parallel circles TU [and] RS in the sphere cut off equal arcs LX [and] XM of great circle BD at the largest of the parallels HQ, [circle] [72b] RS is equal to TU. Now, since the equal and parallel circles RS [and] TU in the sphere cut off arcs TH [and] HR of great circle $ABGD$ at the largest of the parallels HQ, [arc] TH is equal to HR. Also, zH is equal to Hq since LX is equal to XM. Therefore, the [line segment] from q to R is equal to the [line segment] from T to z. And circle RS is equal to circle TU. Therefore, arc qR is equal to arc Tz. But qR is similar to arc He; and, therefore, He is similar to Tz.

Therefore, in the time in which e (beginning from e) traverses arc eH, z traverses zT and comes to T. But arc eR (that is, arc LX) sets while e comes to H; and arc zH (that is, XM) sets while z comes to T. Therefore, arc LX sets in an equal time to arc XM.

Then, similarly, we shall show that KX too sets in an equal time to XN. Therefore, the remaining arc KL sets in an equal time to MN.

Then, similarly, we shall show that BK too sets in a time equal to ND.

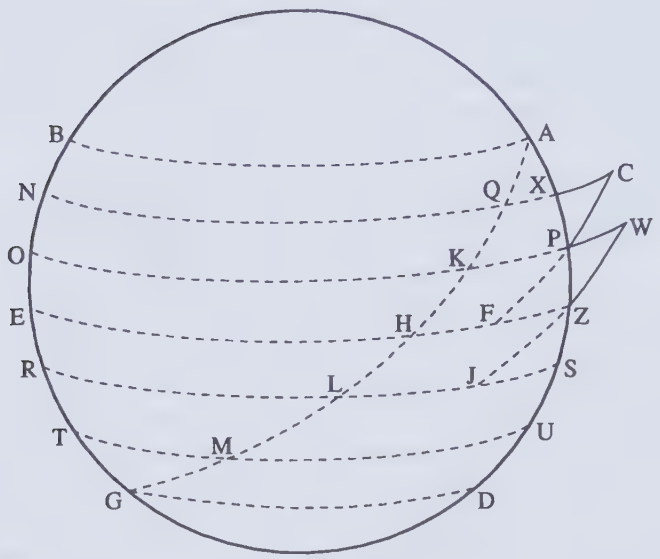

Figure 28. Proposition 12b. Part 2

Since BK sets in a greater time than KL and KL [in a greater time] than LX, but DN [sets] in the time in which BK sets, MN [sets in the time] in which KL sets, and XM [sets in the time] in which LX sets; [it follows that] DN sets in a greater time than NM and NM than MX.

[*Part 2*]. I say also that LX rises in an equal time to XM, KL [rises in an equal time] to MN, and BK to ND.

[74b] Consider what has been said [but applied] to the second figure [Figure 28], and let the semicircle following Cancer below the earth be AG. Let each of AH [and] HG be divided into three equal parts by the points Q [and] K, L [and] M; and let NX, OP, RS, [and] TU be the parallel circles on which Q, K, L, [and] M move.

Since ZH is longer than [an arc] similar to KP, let HF be similar to KP. Then, while K (beginning from K) traverses arc KP and comes to P, H (beginning from H) traverses arc HF and reaches F, and the ecliptic will have position FPC. Again, since LS is longer than [an arc] similar
[76b] to HZ, let HZ be similar to LJ. Then, while H traverses arc HZ and comes to Z, L traverses arc LJ and will reach J, and the ecliptic will have position JZW.

Since the parallel circles OP [and] RS in the sphere cut off equal arcs LH [and] HK of great circle AG at the largest of the parallels EZ, [circle] OP is equal to RS. Now, since the equal and parallel circles PO [and] RS

in the sphere cut off arcs SZ [and] ZP of great circle $ABGD$ at the largest of the parallels EZ, arc SZ is equal to ZP. But JZ is equal to ZW; and, therefore, the [line segment] from P to W is equal to the [line segment] from J to S. And circle OP is equal to circle RS; therefore, arc PW is equal to arc JS.

Since the semicircle from C in the direction of CP is asymptotic to the semicircle from W in the direction of WZ, arc PW is similar to arc FZ. But PW is similar to JS; and, therefore, FZ is similar to JS. Therefore, in the time in which F traverses arc FZ and comes to Z, J traverses JS and comes to S. But, when F comes to Z, arc PF (that is, KH) rises; and, when J comes to S, arc JZ (that is, LH) rises. Therefore, KH rises in an equal time to LH.

Similarly, we shall show that KQ rises in an equal time to LM, and AQ to MG.

Therefore, equal arcs of the semicircle following Cancer set in unequal times, those at the points of contact with the tropics in the greatest [times], those following them in lesser [times], and those at the equator in the least, but those equidistant from the equator both set and rise in equal [times]. [78b]

Proof. In the following $m(XY)$ denotes the angular measure of the arc XY, a concept foreign to Greek mathematics of Euclid's time.

The horizon is $ABGD$; EZ, the arctic circle; BA, the summer tropic; GD, the winter tropic; BD, the semicircle following Cancer, assumed to be above the earth; HQ is the equator; and BK, KL, LX, XM, MN, and ND are the six equal arcs into which the semicircle following Cancer is assumed to be divided, that is the zodiacal signs Cancer, Leo,... Sagittarius. Through the endpoints of these arcs pass the parallel circles OP, RS, and so on. Also, $JKsa$ and WLb are rotated versions of the western horizon, passing through points K and L on the ecliptic. As in **a**, the Equator Arcs Theorem guarantees that $m(Ha) > m(ab) > m(bX)$. On the other hand, given the Asymptotic Semicircles Theorem, $m(Ha) = m(OK)$ and $m(ab) = m(sL)$; so, $m(OK) > m(sL)$. Also, $m(LR) > m(OK)$, since $m(LR) > m(sR) = m(OK)$. Euclid chooses g on RL so that $m(Lg) = m(OK)$. Thus, while K traverses KO, the point L traverses Lg; this means that at some time the ecliptic will have the position gOd. Next, Rs is similar to gL, since both are similar to OK. Since they are arcs of the same circle, $Rs = gL$; accordingly, if we remove the common arc gs (or add it as the case may be), we find Rg equals sL. Hence, $m(OK) > m(Rg)$; thus, it takes K longer to traverse KO than it does g to traverse gR. But, as K traverses KO, arc BK sets; and, as g traverses gR, arc Og sets. But the setting time of Og is equal to that of KL, since, when

K has reached O, L has reached g. Hence, the setting time of BK is larger than that of KL.

There are no new ideas in the proof that KL sets in greater time than LX.

The proof that equal arcs equidistant from the equator set in equal times uses the images qHz, hRe, and dOg of the ecliptic which is rotated so that it passes successively through H, R and O on the western horizon. To these images Euclid applies two tacit assumptions:

(1) given the position of the ecliptic in this proof, namely, BD, the farther south a parallel is the longer is its visible arc between the ecliptic and the western horizon; and

(2) arcs cut off on two parallels by the rotated images of the ecliptic are similar, which is a consequence of the Asymptotic Semicircles Theorem.

Euclid applies (1) explicitly to conclude that, if HX is the arc that the ecliptic cuts off from the equator towards the horizon and TM the arc it cuts off on a southern parallel, then $m(HX) < m(TM)$. He also applies (1) implicitly in the unstated inference that $m(RL) < m(HX)$. Now, Euclid chooses z so that $m(zM) = m(HX)$, which entails that, when the ecliptic passes through H on the horizon, it has the position zHq.

Then, since (by hypothesis) LX equals XM, it follows from the Theorem of Parallels that the parallel circles RLS and TMU are equal; and from this it follows according to the same theorem that HR equals HT. Also, since LX equals XM, the rotated images of these arcs, qH and Hz, are equal. Hence, by *Sphaerica* iii 3,[55] the line segments Rq and zT are equal; and, since these are on equal circles, it follows that arc zT equals arc Rq.

By (2), Rq and He are similar; so, zT and He are similar. Thus, e traverses eH to arrive at H in the same time that z traverses zT to arrive at T; and, so, arcs Re and Hz (and, hence, LX and XM) set in equal times. Similarly, we could show that KX and XN set in equal times, so that the remaining arcs KL and MN also set in equal times. Likewise, BK and ND set in equal times.[56]

Regarding the rising times, the proof in **b** begins with the words, 'Consider what has been said [but applied] to the second figure'; and, indeed, it suffices to reproduce the figure for the Part 2 here, where AHG is now the

[55] Suppose that two great circles in a sphere cut each other and that equal arcs are cut off on each of them from a point at which they cut each other. Then, the straight lines joining the endpoints of the [two pairs of arcs] in the same direction are equal.

[56] Oddly, these statements about the rising times of differences of arcs, which amount to the tacit assumption of the additivity of rising times for disjoint arcs, are proved in Proposition 10 for the special case of semicircular arcs.

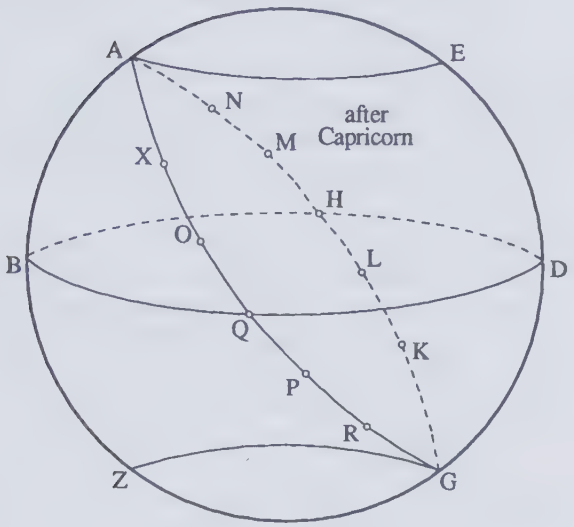

Figure 29. Proposition 13a

semicircle of the ecliptic following Cancer, but now below the horizon with the beginning of Cancer, A, rising. One sees immediately that the previous arguments apply with only a change of lettering to show that KH and HL rise in the same times, as do QK and ML, as well as AQ and GM.

✣ ✣ ✣ ✣ ✣

[Proposition] 13. *Equal arcs*[57] *of the semicircle following Capricorn rise in unequal times, those at the points of contact with the tropics in the greatest [times], those following them in lesser [times], and those at the equator in the least, but those equidistant from the equator both set and rise in equal [times].*[58]

[**a**: Figure 29] In the cosmos, let ABG be the horizon; AE, the summer tropic; GZ, the winter tropic; BD, the equator; let the ecliptic have position $AHGQ$; let GHA be the semicircle following Capricorn below the earth, and let each of the quadrants below the earth, and let GH [and] HA be divided into the signs at K [and] L, M [and] N.

I say that arcs GK, KL, LH, HM, MN, [and] NA rise in unequal times; GK [and] NA in the greatest [time]; KL [and] MN in lesser [times]; [80a]

[57] Actually, only the signs, as the proof indicates.

[58] So **a**: **b** has 'both rise and set in an equal [time]'

LH [and] HM in the least [time]; but [that] GK [rises] in [a time] equal to NA, and KL to MN, and LH to HM.

Let each of the quadrants AQ [and] QG of the semicircle following Cancer be divided into the signs by X [and] O, P [and] R. The circle $[AHGQ]$ will
[82a] then be divided into twelve equal [signs]. [It is] obvious that [arcs] AX, XO, OQ, QP, PR, [and] RG are both equal and opposite to GK, KL, LH, HM, MN, [and] NA.

Since the equal arcs of the semicircle following Cancer do not set in an equal time, but those at the points of contact with the tropics in the greatest, those following them in lesser [times], and those at the equator in the least, but those equidistant from the equator in an equal time [cf. Proposition 12], [it follows that] arcs AX [and] RG set in the greatest time, XO [and] PR in lesser [times], and OQ [and] OP in the least [time], and [that] AX [sets] in an equal [time] to RG, and XO to PR, and OQ to QP. But, while AX, XO, OQ, QP, PR, [and] RG set, GK, KL, LH, HM, MN,
[84] [and] NA rise. Therefore, GK [and] NA rise in the greatest time, KL [and] MN in lesser [times], and LH [and] HM in the least [time], and AN [rises] in an equal time to GK, and MN to KL, and HL to MH.

Since QP also [rises] in the time in which OQ rises, but LH sets in the time in which OQ rises, and HM sets in the time in which QP rises, [it follows that] HL sets in an equal time to HM. Then, for the same [reasons], MN [sets in an equal time] to LK, and NA to KG.

[80b] [**b**: Figure 30] Let $ABGD$ be the horizon; BA, the summer tropic; [and] GD, the winter topic. Let DHB be the semicircle following Capricorn below the earth; $EQHZ$, the equator; and let each of BH [and] HD be divided into three equal [parts] at the points K [and] L, M [and] N.

I say that BK, KL, LH, HM, MN, [and] ND rise in unequal times; BK [and] ND in the greatest [time]; KL [and] MN in lesser [times]; LH [and] HM in the least [time]; but [that] BK rises and sets in an equal [time] to ND, and KL to MN, and LH to HM.

[82b] [*Proof*]. Let BQD be the semicircle following Cancer above the earth, and let each of BQ [and] QD be divided into three equal [signs] by the [points] O [and] P, R [and] S.

Since BO sets in a greater time than OP, but DN rises while BO sets, and MN rises while OP sets, [it follows that] ND rises in a greater time than MN. Again, since OP sets in a greater time than PQ, but OP sets while NM rises, and PQ sets while HM rises, [it follows that] NM rises in a greater time than MH. Then, for the same [reasons], BK too rises in a greater time than KL, and KL than LH.

Also, since QR [sets] in the time in which PQ sets, but MH rises while PQ sets, and HL rises while QR sets, [it follows that] MH rises in an equal

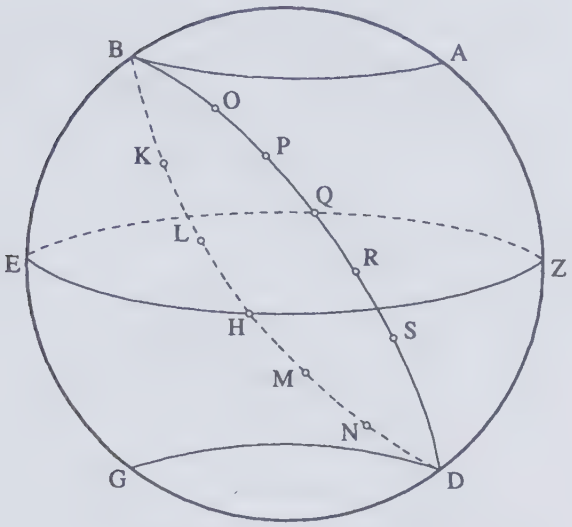

Figure 30. Proposition 13b

time to HL. Then, for the same [reasons], KL rises in an equal time to MN, and BK to DN.

Again, since QR [rises] in the time in which PQ rises, but MH sets while PQ rises, and HL sets while QR rises, [it follows that] LH sets in an equal time to HM. Then, for the same [reasons], KL sets in an equal time to MN, and BK to DN.

Paraphrase. Equal arcs of the semicircle following Capricorn rise in unequal times, those at the points of contact with the tropics in the greatest time, then those following them in less time, and those at the equator in the least time, but those equidistant from the equator set and rise in equal times.

Proof. The proof, given in both **a** and **b**, is the obvious one, namely, an application of Proposition 11 to reduce the proposition to one about the setting times of arcs on the semicircle following Cancer exactly as in Proposition 12. Euclid carries the argument through in painful detail, arc by arc.

<Lemma.[59] *Of equal arcs of the ecliptic that are equidistant from the point of contact with either tropic, in the time in which one rises the other sets and* vice versa.

[59] So Menge's text. It is labelled 14 in some mss of **a** and is a scholium in **b**.

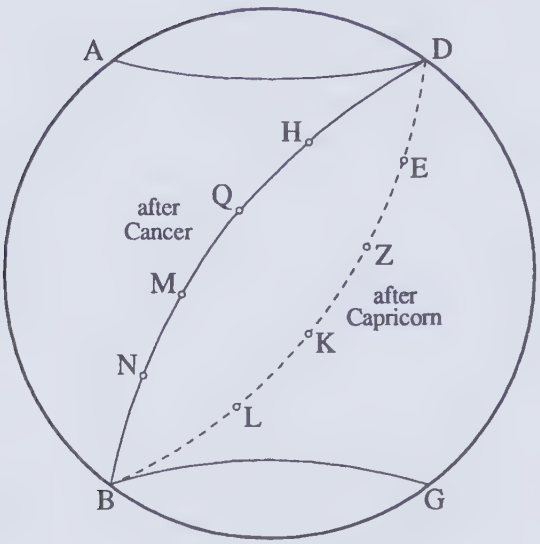

Figure 31. The Lemma

In the cosmos [Figure 31], let $ABGD$ be the horizon; AD, the summer [tropic]; BG, the winter [tropic]; and let the ecliptic have position BED. Let arcs EZ [and] HQ be chosen to be equal and equidistant from the point of contact with the tropic.

I say that HQ sets [in the time] in which EZ rises.

Let it be assumed that semicircle BQD following Cancer is setting. And let arc KL be chosen equal and opposite to HQ on the semicircle following Capricorn.

Then, KL [and] EZ, being equidistant from the equator and from the points of contact with the tropic, rise in an equal time. But, while KL rises, the opposite [arc] QH sets. Therefore, QH sets [in the time] in which EZ rises and *vice versa*, the very [thing it was] necessary to show.>

Paraphrase. Of equal arcs of the ecliptic that are equidistant from its point of contact with either tropic, the rising time of one is the setting time of the other.

Remark. This result serves as a lemma for Part 2 of Proposition 14 and, hence, introduces the last topic of the treatise, the time of passage of an arc over the visible or invisible hemisphere. Propositions 14–18 are concerned exclusively with this topic.

Proof. Let EZ and HQ be arcs satisfying the hypotheses. If KL is opposite HQ, then, by Proposition 11, the rising time of either of HQ and KL is the setting time of the other. But KL and EZ rise and set in equal times, by Proposition 12. Thus, the rising time of either of HQ and EZ is equal to the setting time of the other.

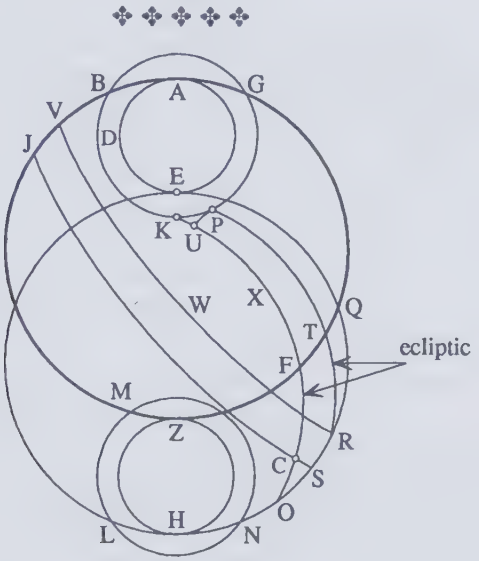

Figure 32. Proposition 14a. Part 1

[Proposition] 14 [a] [Part 1]. Equal arcs of the ecliptic [in the semicircle preceding Cancer] do not pass across the visible hemisphere in an equal time, but the one nearer the summer tropic always in a greater [time] than the [one] farther off, whenever the zenith is between the [observer's] arctic [circle] and the summer tropic. [86a]

In the cosmos [Figure 32], let ABG be the horizon; AD, the largest of the ever-visible [circles]; ZH, [the largest] of the ever-invisible [circles]; BGK, the summer tropic; and LMN, the winter [tropic]. Let the ecliptic have position KXO at one time and PTR at another; [let U be the intersection of KXO and PTR above the earth;][60] and let the arc KO be cut off not longer than a semicircle. [88a]

[60] Point U is mentioned later without definition. Although the letter U appears in Menge's diagram, accurately reflecting recension **a** of cod. Vind. gr. 31.13, the U in the diagram has no point to which it obviously refers; but it is in the right place to label the intersection of RTP produced and KF.

Through E let there be drawn a great circle touching ADE; then, it touches ZH too. Then, either it will go through K or it will fall beyond K in the direction of B[, say] (for it makes no difference).

Let it [the great circle] be EHQ, and let the semicircle from E in the direction of R be asymptotic to the semicircle from A in the direction of B.

Now, since the pole of circle ABG [the zenith] is between ADE [and]
[90a] BKG, and circles [KXO and PTR] have been drawn to intersect BAG, arc UF is shorter than UPT. Again, since the pole of circle ABG is between ADE [and] BKG, and ERO touches ADE, the pole of ERO is also between ADE [and] BKG (therefore, its other pole is between ZH [and] MNL). Therefore, arc UFO is longer than UPR. But arc UF is shorter than UPT; therefore, the remaining [arc] OF [of UFO] is longer than RT [the remainder of UTR].

Let similar and equal arcs TR [and] FC be laid off [that is, FC similar and equal to TR], and let the parallel circles on which points R [and] C move be JCS [and] VWR; then, JCS [and] VWR are similar.

[92a] [In the time] in which S traverses arc SCJ, R traverses arc RWV. Therefore, R traverses arc RWV in a greater time than C [traverses] JC. But RT passes across the visible hemisphere while R traverses RWV, and FC passes across the visible hemisphere while C traverses CJ. Therefore, TR passes across the visible hemisphere in a greater time than FC.

[Part 2. Figure 33]. *In like manner also equal arcs in the other semicircle [of the ecliptic] do not pass across the visible hemisphere in an equal time, but always the [one] nearer the point of contact with the summer tropic in a greater time than the [one] farther off. But those [equal arcs] equidistant from the point of contact in either [semicircle pass across] in an equal [time].*

In the cosmos, let ABG be the horizon; AEZ, the summer tropic; and let
[94a] the ecliptic have position BEG; and let equal arcs KH [and] HM be cut off [on it].

I say that KH passes across the visible hemisphere in a greater time than HM.

[96a] [*Proof*]. Let there be drawn the parallel circles KL, HQ, [and] MN on which K, H, [and] M move; then, the arcs KH [and] LQ are equal and equidistant from either point of contact [of the ecliptic with a tropic].

Since arc KH sets [in the time] in which LQ rises, but also, [in the time] in which Q (beginning from X) traverses XQO, [point] H (beginning from X) traverses XQO, [it follows that] the time in which LQ rises added to the time in which Q (beginning from X) traverses XQO is equal to the time in which H (beginning from X) traverses XHO added to the time in which HK sets. But the time in which LQ rises added to the time in

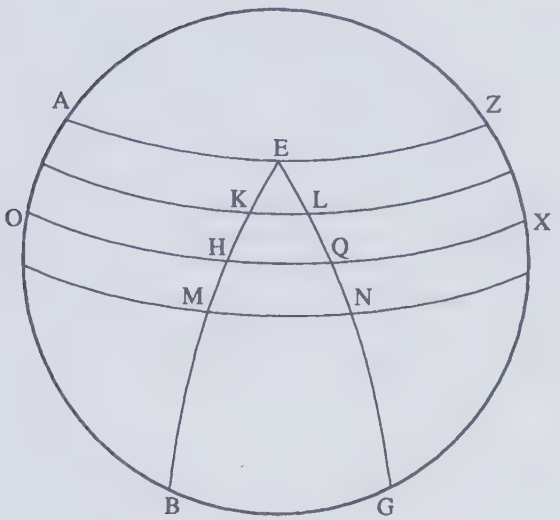

Figure 33. Proposition 14a. Part 2

which Q (beginning from X) traverses arc XQO is the time in which LQ [98a] passes across the visible hemisphere. And the time in which H (beginning from X) traverses XHO added to the time in which KH sets is the time in which KH passes across the visible hemisphere. Therefore, LQ [and] KH pass across the visible hemisphere in an equal time. And QN and HM pass across the visible hemisphere in an equal time for the same [reasons].

But LQ passes across the visible hemisphere in a greater time than QN (for it was shown in the previous part [of the proposition]); therefore, KH passes across the visible hemisphere in a greater time than HM. And, at the same time, it was shown [in this part of the proposition] that [arcs] equidistant from the point of contact in either [semicircle] pass across the visible hemisphere in an equal time.

✣ ✣ ✣ ✣ ✣

Paraphrase. Suppose $\epsilon < \phi < 45°$. Of two equal arcs of the ecliptic, the one nearer the summer tropic takes more time to pass across the visible hemisphere than does that farther off; but, if they are equidistant from the summer tropic, they take the same time [cf. p. 32, above].

Remarks. (1) In our paraphrase we have chosen the highly unsatisfactory (because so unfaithful to Euclid's outlook) '$\epsilon < \phi < 45°$', since the more faithful 'when the zenith is north of the summer tropic but further south of the pole than is the nearest point on the horizon' is awkward.

(2) The hypotheses of this proposition are, geographically, the most restrictive in the work, but the range of latitudes (from Aswan—ancient Syene—to the estuary of the Danube) does include most of the world Euclid would have been familiar with. The reason that the zenith cannot be between the tropics is that then arcs near the summer tropic could take a shortcut over the visible hemisphere by going behind (that is, to the north of) the zenith. For example, for localities on the equator it is obvious that a 10° arc of the ecliptic centred on the equator will take longer to cross the visible hemisphere than does the last 10° ending at a tropic. Further, putting the condition another way, if the summer tropic were always visible, some arcs of the ecliptic near it would always be in the visible hemisphere, which means that the idea of their time of crossing the hemisphere would not make sense. Thus, it is clear that $\phi < 90° - \epsilon$.

Why, however, ϕ must be less than 45°, as the condition that the visible pole lies outside of the arctic circle requires, is not at all clear, other than that this condition is necessary for the proof of *Sphaerica* ii 22, which is in turn useful in proving the latter conclusion of the Inclined Arcs Theorem.

(3) By checking on an astrolabe or solid sphere, one finds that, of two points in the semicircle following Capricorn, the one nearer Capricorn rises first; and, similarly, that, if the two points are chosen in the semicircle following Cancer, then the one nearer Cancer rises first. Concerning the diagram on Menge 1916, 86 [see our Figure 32], one can now reason backwards and conclude that, since according to the conclusion TR crosses the horizon in greater time than the equal arc FC, the arc TR is nearer to the summer tropic than is FC; and this fits with the visual impression from the diagram that PT is shorter than KF. If this is the diagram for arcs in the semicircle following Capricorn, then FC must rise first, since it is nearer Capricorn than TR. And, TR will appear on the horizon to the north of where FC appeared, which again agrees with the diagram. The only thing the reader must remember is that PTR is a subsequent position of the ecliptic.

(4) Finally, as the closing remarks in our commentary on the introduction indicate, this proposition is of interest when the arcs considered are very small, that is, when they are on the order of a degree or so.

Proof. Implicit in the proof is the assumption that we are dealing first with the semicircle of the ecliptic following Capricorn. With reference to Figure 32, consider two equal arcs of this semicircle, say FC and TR, one from each of two different positions of this semicircle, and let TR be the one nearer the summer tropic. Consider EQR, the rotated image of the horizon ABG passing through R and tangent to the arctic circle at, say, E. Of the two possible such circles passing through R, choose the one so that the semicircle containing EQR is a rotated version of the horizon semicircle

ABZ. Let it cut the first position of the ecliptic, KFC, at O. According to the Inclined Arcs Theorem when applied to the two ecliptic positions UFO and PTR, arc UF is shorter than UPT, and UPR is shorter than OFU. If we remove UF from OFU and UPT from UPR, that is, the shorter from the longer, we find TR is shorter than OF. Thus C, the other endpoint of the arc FC, which is equal to TR but further from the summer tropic, must be between F and O. Now, consider the parallels, RWV and SCJ through the endpoints of the two equal arcs. In light of the Asymptotic Semicircles Theorem,[61] $m(RWV) = m(SCJ)$; so, R will traverse RWV in the same time that S traverses SCJ and, hence, in greater time than C traverses CJ. But, as R traverses RWV, the arc TR passes across the visible hemisphere; and, as C traverses CJ, the arc CF passes across the visible hemisphere. So, the proposition is proved for arcs of the semicircle of the ecliptic following Capricorn.

Euclid proves Part 2 by showing first that arcs of the ecliptic between the same parallels pass across the visible hemisphere in the same time. Then, he uses this result and the first part to prove the result stated for equal arcs of the ecliptic in the semicircle following Cancer. In Figure 33, we are looking at the part of the ecliptic that has moved so that its northern point, E, is above the visible hemisphere. Comments in the proof make it clear that ZX is part of the eastern horizon and AO part of the western. Hence, $ELQN$ is part of the semicircle following Cancer, and $EKHM$ is part of the semicircle following Capricorn. Let parallel circles KL and HQ be drawn, so that, by the Theorem of Parallels, KH and LQ are equal arcs of the ecliptic that are equidistant from the summer tropic. (They are not assumed to be in the visible hemisphere, even though the diagram shows them there.) Hence, Q rises after L but H rises before K—both of which facts are tacitly assumed. (Since the hypotheses of this proposition imply that the hypotheses of Proposition 4 apply to the ecliptic, we may conclude that Q also sets after L and H sets before K.) Also, by the Lemma preceding the proposition, the rising time of LQ equals the setting time of KH. Further, H and Q cross the visible hemisphere in equal times, since they are on the same parallel. But the time it takes LQ to pass across the visible hemisphere is the sum of the rising time of LQ and the time it takes Q to cross the visible hemisphere, while the time it takes HK to pass across the visible hemisphere is the time it takes HK to set added to the time it takes H to cross the visible hemisphere. Since these sums are equal, it follows that KH and LQ pass across the visible hemisphere in equal times. Similarly, if MN is another parallel, HM and QN pass across in equal times. But, since LQ passes across in

[61] We explain this notation in our commentary [p. 91, above] to the proof of Proposition 12 found in **b**.

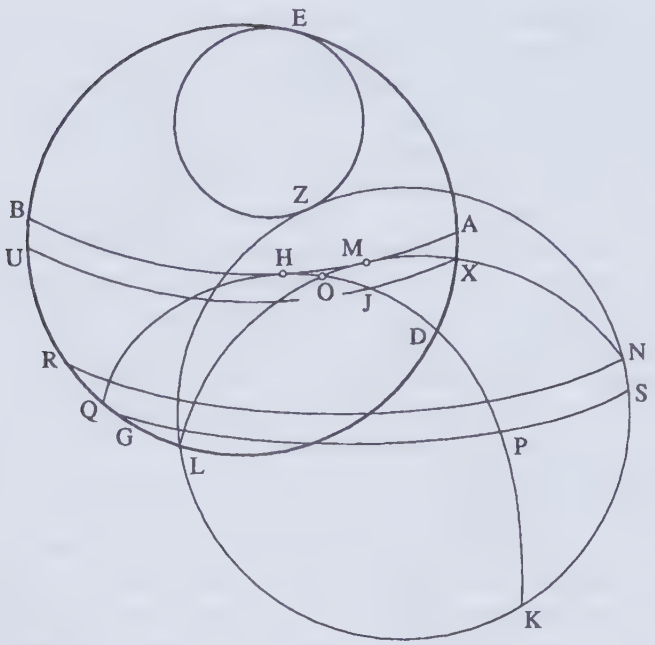

Figure 34. Proposition 14b. Part 1

a greater time than QN, it follows that KH passes across in a greater time than HM. Consequently, all parts of the proposition have been established.

[86b] [Proposition] 14 [b] [Part 1. Figure 34]. *Equal arcs of the ecliptic do not pass across the visible hemisphere in equal times, but the one nearer the point of contact with the summer tropic in a greater time than the [one] farther off, whenever the zenith is between the arctic [circle] and the summer tropic.*

[88b] Let $ABGD$ be the horizon; EZ, the largest of the ever-visible [circles]; and BA, the summer tropic. Let the pole of $ABGD$ [that is, the zenith] be between EZ [and] BA, and let the ecliptic have position QHK at one time and LMN at another. Let arc HK be cut off not longer than a semicircle; let the great circle KNZ be drawn through the point K touching EZ; [and let O be the intersection of HK and LM].

Since the great circle $ABGD$ in the sphere touches circle EZ, but intersects a different circle BA parallel to it [EZ], and [since] the pole of $ABGD$ is between AB [and] EZ, and [since] the great circles QHK [and] LMN have been drawn touching [circle] BA, [it follows that] arc OMX

is longer than arc OD. Again, since the great circle $ABGD$ in the sphere touches the circle EZ, but intersects a different circle BA parallel to it, and [since] the pole of circle $ABGD$ is between BA [and] EZ, and [since] the great circle ZNK has been drawn touching EZ, [it follows that] the pole of circle ZNK is between EZ [and] BA. Therefore, its other pole is between the [circles] equal and parallel to EZ [and] BA. Therefore, arc KO is longer than OMN, [and] on them XMO is longer than OD. Therefore, the remaining [arc]DK is longer than XN. [90b]

Let DP be laid off equal to NX, and let NR [and] GPS be the parallel circles on which points N [and] P move.

Since the semicircle from E in the direction of ER is asymptotic to the semicircle from Z in the direction of ZN, arc NR is similar to arc GS; therefore, NR is longer than [an arc] similar to GP. Therefore, N (beginning from N) traverses arc NR and comes to R in a greater time than P (beginning from P) traverses arc PG and comes to G. But, while N traverses the arc NR and comes to R, [arc] NX passes across the visible hemisphere; and, while P (beginning from P) traverses PG and comes to G, [arc] PD passes across the visible hemisphere. Therefore, XN passes across the visible hemisphere in a greater time than DP.

(I say also that XN is nearer the point of contact with the summer tropic than PD.

Let there be drawn through X a parallel circle XU. Then, arc XM is equal to HJ; therefore, HD is longer than MX. Therefore, XN is nearer to the point of contact with the summer tropic than PD.)

[Part 2. Figure 35]. *In like manner also equal arcs in the other semicircle [of the ecliptic] pass across the visible hemisphere in unequal times, the [one] nearer the point of contact with the summer tropic in a greater [time] than the [one] farther off. But, in each of the semicircles, those [equal arcs] equidistant from the summer tropic [pass across] in an equal time.* [92b]

Let $ABGD$ be the horizon; EZ, the greatest of the ever-visible [circles]; BHA, the summer tropic; and let the ecliptic have position GHD.

I say that, in the other semicircle in the direction of HG, equal arcs do not pass across the visible hemisphere in equal times, but [that] the [one] nearer the point of contact with the summer tropic [passes across] in a greater [time] than the [one] farther off. But those equidistant from the summer tropic in each of the semicircles [pass across] in an equal time.

Let the parallel circle DQ be drawn. Then, KH is equal to HD. Let the ecliptic be rotated and have position QLR. Since KH [and] HD are equidistant from the point of contact with the summer tropic, KH (that is, LQ) sets in the time in which DH rises. But the time in which DH rises [94b]

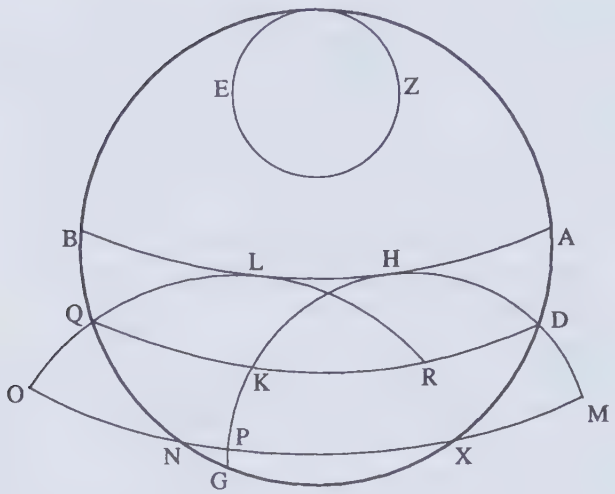

Figure 35. Proposition 14b. Part 2

is the time in which H (beginning from A) traverses arc AH and comes to H; and the time in which LQ sets is the time in which L (beginning from L) traverses arc LB and comes to B. Therefore, in the time in which H traverses AH and comes to H, L too traverses arc LB and comes to B.

Let the common time be added in which D (beginning from D) traverses arc $DRKQ$ and comes to Q. Then, the time in which H (beginning from A) traverses arc AH and comes to H plus the time in which D (beginning from D) traverses arc DQ and comes to Q is equal to the time in which L (beginning from L) traverses arc LB and comes to B plus the time in which Q (beginning from D) traverses arc DKQ and comes to Q. But the time in which H (beginning from A) traverses arc AH and comes to H plus the time in which D (beginning from D) traverses arc DQ and comes to Q is the time in which arc HD passes across the visible hemisphere; and the time in which L (beginning from L) traverses arc LB and comes to
[96b] B plus the time in which Q (beginning from D) traverses arc DRQ and comes to Q is the time in which LQ (that is, KH) passes across the visible hemisphere. Therefore, HD too [passes across] in the time in which arc KH passes across the visible hemisphere.

Then, let the point M be chosen so that HD is equal to DM, and let $MXNO$ be the parallel circle on which point M moves. Then, DM is equal to KP, and DM [and] KP are equidistant from the point of contact with the summer tropic. Therefore, arc PK (that is, QO) sets in the time in which arc DM rises. But the time in which DM rises is the time in

which M (beginning from M) traverses arc MX and comes to X. And the time in which QO sets is the time in which O (beginning from N) traverses arc NO and comes to O. Therefore, the time in which M (beginning from M) traverses arc MX and comes to X is the same as the time in which O (beginning from N) traverses NO and comes to O.

Let the common time be added in which X (beginning from X) traverses XN and comes to N. Then, the time in which M (beginning from [98b] M) traverses arc MN and comes to N is equal to the time in which O (beginning from X) traverses arc XO and comes to O. But the time in which M (beginning from M) traverses arc MN and comes to N is the time in which arc DM passes across the visible hemisphere. And the time in which O (beginning from X) traverses arc XO and comes to O is the time in which OQ (that is, KP) passes across the visible hemisphere. Therefore, KP too [passes across] in the time in which DM passes across the visible hemisphere.

And, since HD passes across the visible hemisphere in a greater time than DM, but [since] HK too [passes across] in the time in which HD passes across the visible hemisphere, and [since] KP also [passes across in the time] in which DM [passes across the visible hemisphere], [it follows that] HK passes across the visible hemisphere in a greater [time] than KP.

The proof of Part 1 in **b** is confused in that it appears to demonstrate both the conclusion and the hypothesis; but the idea in the proof of the conclusion is the same as in **a** and both versions depend on the Inclined Arcs Theorem. In our translation of **b** we have put the text's proof of the hypothesis in parentheses.

The proof of Part 2 in **b** is based on the same ideas as in **a**; but these ideas are applied to demonstrate, with reference to Figure 33, that EL and EK pass across the visible hemisphere in equal times, as do LQ and KH. But, since EL passes across in greater time than LQ, it follows that EK passes across in greater time than KH.

[Proposition] 15. Of equal and opposite arcs of the ecliptic, while one passes [100a] across the visible hemisphere, the other [passes across] the invisible; and, while the one passes across the invisible [hemisphere], the other [passes across] the visible.

[**a**: Figure 36] In the cosmos, let ABG be the horizon; let the ecliptic have position $ZAEG$; and let equal and opposite arcs AE [and] GZ be cut off [on it].

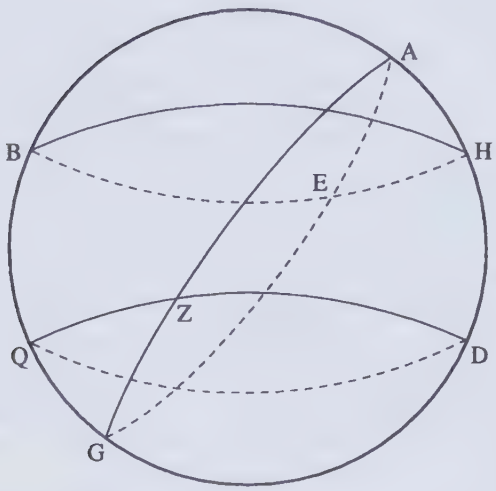

Figure 36. Proposition 15a

I say that, while AE passes across the visible hemisphere, GZ [passes across] the invisible [hemisphere]; and, inversely, that, while AE passes across the invisible hemisphere, GZ [passes across] the visible.

[102a] [*Proof*]. Let AEG be the semicircle below the earth, and let EHB [and] ZQD be parallel circles on which the points E [and] Z move.

Since diametrically opposite stars on the ecliptic rise and set in syzygy, when E sets at B, [point] Z (diametrically opposite to it [E]) rises at D. Therefore, while E traverses arc EHB and comes to B, Z traverses ZQD and comes to D. But the time in which E traverses arc EHB and comes to B is the time in which AE passes across the visible hemisphere; and the time in which Z traverses ZQD is the time in which GZ passes across the invisible hemisphere. Therefore, while AE passes across the visible hemisphere, GZ [passes across] the invisible.

[100b] [**b**: Figure 37] In the cosmos, let $ABGD$ be the horizon; let AD be the summer tropic; BG, the winter tropic; let the ecliptic have position $DEBZ$; let DEB be the semicircle following Cancer below the earth; BZD, the [semicircle] following Capricorn above the earth; let D be the eastern side and B the western; and let two equal and opposite arcs DE [and] BZ be cut off [on the semicircles].

I say that, while DE passes across the visible hemisphere, ZB [passes across] the invisible; and that, while DE [passes across] the invisible, BZ [passes across] the visible.

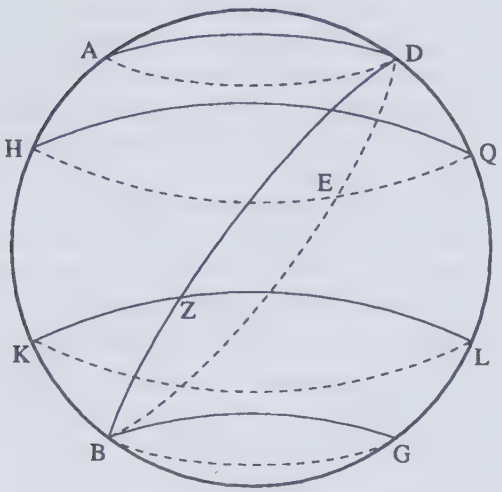

Figure 37. Proposition 15b

[*Proof*]. Let there be drawn parallel circles HEQ [and] KZL on which points E [and] Z move. [102b]

Since diametrically opposite stars on the ecliptic rise and set in syzygy, when point E sets at point H, [point] Z (diametrically opposite to it [E]) rises at point L. But E sets when it has traversed EQH, and Z rises when it has traversed ZKL. Therefore, Z [traverses] ZKL while E traverses arc EQH. But the time in which E traverses EQH is the time in which DE passes across the visible hemisphere; and the time in which Z traverses ZKL is the time in which ZB passes across the invisible hemisphere. Therefore, DE passes across the visible hemisphere and ZB [passes across] the invisible [hemisphere] in an equal time.

Similarly, we shall show also that while DE passes across the invisible hemisphere, ZB [passes across] the visible.

Paraphrase. Of equal and opposite arcs of the ecliptic, while one passes across the visible hemisphere, the other passes across the invisible; and, while the one passes across the invisible hemisphere, the other passes across the visible.

Proof. The proof in **a** begins with the suppositions that arc AE in the invisible hemisphere has A just rising and that the equal and opposite arc GZ in the visible hemisphere has G just setting. (Since the endpoints of the arcs are diametrically opposite, it follows from Proposition 6 that, when A is rising, G is setting.) Now, the time in which AE passes across the

visible hemisphere is the time in which E traverses EHB; and the time in which ZG passes across the invisible hemisphere is the time in which Z traverses ZQD. Since the tropic circles are tacitly assumed to be larger than the arctic circle—otherwise, some arcs of the ecliptic would always be visible—Proposition 4 guarantees that, of the two points Z and G on the ecliptic, since G sets first, it also rises first and, consequently, ZG will not have passed across the invisible hemisphere until Z rises. However, since E and Z are also diametrically opposite, when E is setting at B, the point Z is rising at D. Hence, the time in which AE passes across the visible hemisphere is the time in which ZG passes across the invisible hemisphere.

Apart from the lettering, the proof in **b** is the same as in **a**.

Note. In the rest of the treatise, all proofs the are taken from **b**, since **a** contains only a fragment of the proof of Proposition 16 and lacks even the statements of the following propositions.

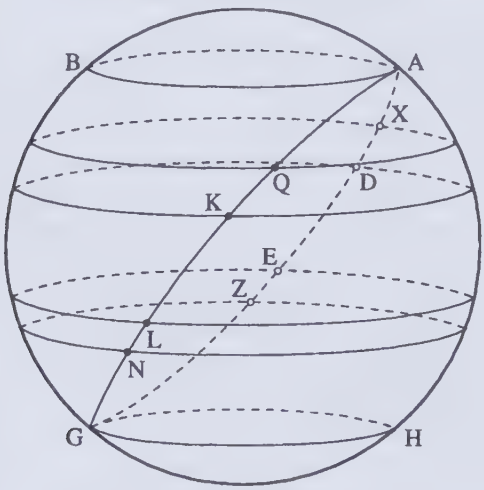

Figure 38. Proposition 16a

[104] [a] [Proposition] 16. *Equal arcs of the ecliptic do not pass across the invisible hemisphere in an equal time, but the [one] nearer the point of contact with the winter tropic always [passes across] in a greater [time] than the [one] farther off, and those [equal arcs] equidistant from either point of contact [with a tropic pass across] in an equal [time].*

In the cosmos [Figure 38], let ABG be the horizon; let the ecliptic have position AZG; and let equal arcs DE [and] EZ be cut off [on it].

[*Part 1*]. I say that arcs DE [and] EZ do not pass across the invisible hemisphere in an equal time, but that ZE [does so] in a greater [time] than ED.

[*Proof*]. Let there be laid off arc QK which is both equal and opposite to ZE, and [arc] KL which is both equal and opposite to ED; then, QK is equal to KL.

Since KQ passes across the visible hemisphere in a greater time than KL... [end of recension **a**]

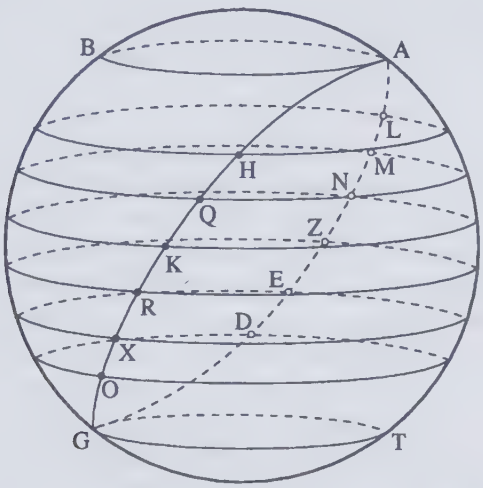

Figure 39. Proposition 16b

[**b**] [Proposition] 16. *Equal arcs of the ecliptic do not pass across the invisible hemisphere in an equal time, but the [one] nearer the winter tropic always [passes across] in a greater time than the [one] farther off, and those [equal arcs] equidistant from either of the points of contact [with a tropic pass across] in an equal time.* [106]

In the cosmos [Figure 39], let ABG be the horizon; let AB be the summer tropic; GT, the winter tropic; let the ecliptic have position AGE; and let equal arcs DE [and] EZ be cut off [on it].

[*Part 1*]. I say that arcs DE [and] EZ do not pass across the invisible hemisphere in an equal time, but that arc DE [passes across] in a greater time than arc EZ.

[*Proof*]. Let arcs HQ [and] QK be cut off both equal and opposite to arcs DE [and] EZ. Then, arcs HQ [and] QK do not pass across the visible hemisphere in an equal time, but HQ in a greater [time] than KQ. But, while HQ passes across the visible hemisphere, DE [passes across] the invisible; and, while QK passes across the visible hemisphere, EZ [passes across] the invisible. Therefore, arcs DE [and] EZ do not pass across the invisible hemisphere in an equal time, but DE in a greater [time] than EZ. [108]

[*Part 2*]. I say also that [arcs] equidistant from either of the points of contact with the tropic pass across the invisible hemisphere in an equal time.

[*Proof*]. Let there be parallel circles DO, EX, ZR, KN, QM, [and] HL on which points D, E, Z, K, Q, [and] H move. Then, the arcs HQ [and] LM pass across the visible hemisphere in an equal time. But, while HQ passes across the visible hemisphere, DE passes across the invisible; and, while LM passes across the visible hemisphere, XO passes across the invisible. Therefore, the arcs ED [and] OX pass across the invisible hemisphere in an equal time.

Paraphrase. Equal arcs of the ecliptic pass across the invisible hemisphere in equal times only if they are equidistant from the points of contact with the winter tropic (and, of course, the summer tropic). And, if they are not equidistant, the one nearer the winter tropic takes longer.

Remark. The text and diagrams assume that the equal arcs not equidistant from the tropics are contiguous, but the proof does not require this.

Proof. Euclid uses Proposition 15 to apply what is proved for the visible hemisphere in Proposition 14, in order to demonstrate the conclusion desired for the invisible hemisphere in this proposition.

Since Proposition 14 is stated only for latitudes $\epsilon < \phi < 45°$ and the proofs of Propositions 16–18 evidently depend on it, it follows that the statements of Propositions 16–18 are incomplete and need the condition stated in the earlier proposition about the position of the zenith.

[Proposition] 17. *Of [two] equal arcs [of the ecliptic] on either side of the equator and equidistant from the equator, in the time in which one passes across the visible hemisphere, the other [passes across] the invisible [hemisphere]; and, in the time in which the one passes across the invisible hemisphere, the other [passes across] the visible.*

In the cosmos [Figure 40], let ABG be the horizon; BDG, the equator; let the ecliptic have position AEQ; and let QK [and] ZH be both equal and equidistant arcs on either side of the equator BDG.

[*Part 1*]. I say that, in the time in which QK passes across the visible hemisphere, ZH [passes across] the invisible.

[110] [*Proof*]. Let arc LM be chosen [so that it is] both equal and opposite to ZH. Then, arcs ML [and] QK pass across the visible hemisphere in an equal time. But, while ML passes across the visible hemisphere, ZH

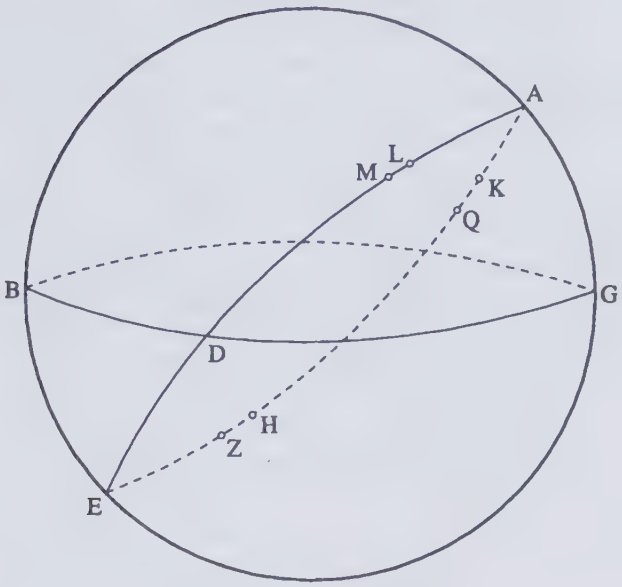

Figure 40. Proposition 17

passes across the invisible. Therefore, in the time in which arc QK passes across the visible hemisphere, arc HZ passes across the invisible.

[*Part 2*]. Then, for the same [reasons], in the time in which arc QK passes across the invisible hemisphere, HZ passes across the visible.

Paraphrase. Of equal arcs of the ecliptic on either side of the equator and equidistant from it, in the time in which one passes across the visible hemisphere, the other passes across the invisible; and, in the time in which the first passes across the invisible hemisphere, the second passes across the visible.

Proof. This proposition follows immediately from Propositions 14 and 15.

[Proposition] 18

[*Part 1*]. *Of any*[62] *two equal arcs in the semicircle [of the ecliptic] toward the summer tropic cut off by the equator, the one of them [nearer the summer tropic] passes across the visible hemisphere in a greater time than the other [passes across] the invisible [hemisphere].*

[62] Not just signs: compare Propositions 12 and 13.

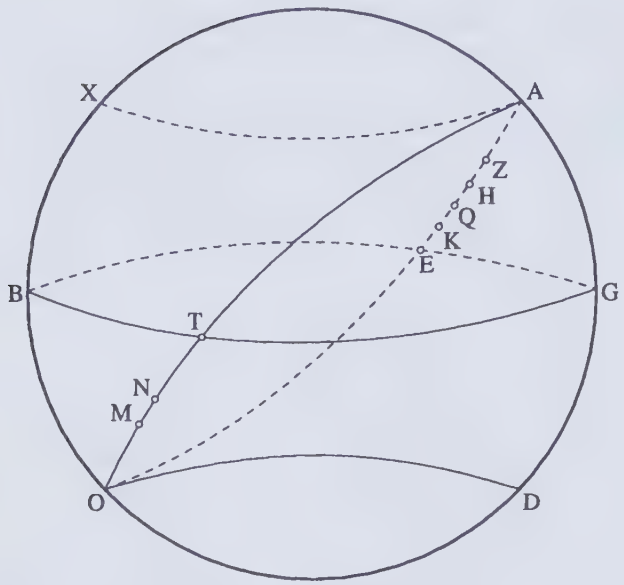

Figure 41. Proposition 18

In the cosmos [Figure 41], let ABG be the horizon; let AX be the summer tropic; DO, the winter [tropic]; let BEG be the equator; and let the ecliptic have position AEO. Let ZH [and] QK be equal arcs in semicircle TAE, and ZH be nearer to the summer tropic.

I say that any arc such as HZ passes across the visible hemisphere in a greater time than any arc such as QK [passes across] the invisible.

Proof. Let arc MN be chosen both equal and opposite to arc QK. Then, ZH is nearer to the summer tropic than MN. Therefore, ZH passes across the visible hemisphere in a greater time than MN [passes across]
[112] the visible. But, while arc MN passes across the visible hemisphere, QK [passes across] the invisible. Therefore, arc ZH passes across the visible hemisphere in a greater time than QK [passes across] the invisible.

Then, similarly, we shall show that any such [arc] passes across the visible hemisphere in a greater time than another such [arc passes across] the invisible.

[*Part 2*]. *And, in like manner, of any [two] equal arcs in the other semicircle [of the ecliptic] toward the winter tropic cut off by the equator, the one of them [nearer the winter tropic] passes across the invisible hemisphere in a greater time than the other [passes across] the visible [hemisphere].*

Paraphrase. [Part 1] Of any equal arcs in the northern half of the ecliptic, the one nearer the summer tropic passes across the visible hemisphere in a greater time than the other passes across the invisible. [Part 2] And, similarly, of equal arcs in the southern half of the ecliptic, the one of them nearer the winter tropic passes across the invisible hemisphere in a greater time than the other passes across the visible.

Proof. Take equal arcs ZH and QK in the northern half of the ecliptic, and suppose HZ is nearer the summer tropic. Pick MN equal and opposite to QK. Since MN is in the southern half of the ecliptic, it is further from the summer tropic than HZ; so, it follows from Proposition 14 that MN traverses the visible hemisphere in less time than does HZ. But the time in which MN traverses the visible hemisphere is, by Proposition 15, the same as the time in which QK traverses the visible. Hence, QK traverses the visible hemisphere in less time than HZ.

The statement in Part 2 follows immediately from that in Part 1 and Proposition 15.

ENGLISH GLOSSARY OF SELECTED TECHNICAL TERMS AND PHRASES

Words in small capitals are themselves entries in this glossary. Our explanations concern Euclid's usage in the *Phaenomena* only. We indicate the location (if any) in Menge 1916 where the Greek rendered by each term is first used, in brackets at the end of its entry in the format [page-number.line-number]. Numbers not in brackets designate pages in our translation where the English equivalent of the Greek term first occurs.

Alternate arcs on the CELESTIAL SPHERE: arcs diametrically opposite—relative to the centre of the sphere—point by point. [4.17] 45

Arctic circle: largest of the EVER-VISIBLE CIRCLEs on the CELESTIAL SPHERE, dependent on latitude of observer. Not to be confused with the modern geographical arctic circle on the surface of the Earth. [2.21] 44

Aquarius: constellation and SIGN of the ZODIAC, the Water Carrier. [12.3] 53

Aries: constellation and SIGN of the ZODIAC, the Ram. [6.25] 47

Asymptotic SEMICIRCLEs *of* GREAT CIRCLES *of the* CELESTIAL SPHERE: semicircles not cutting each other, non-intersecting semicircles. (Neugebauer devotes a section [1975, 758–759] to this term under the label, 'non-intersecting') [22.11] 57

Axis: DIAMETER of the CELESTIAL SPHERE about which this sphere rotates, joining the POLEs of the EQUATOR. [6.13] 46

Cancer: constellation and SIGN of the ZODIAC, the Crab. [10.17] 52

Capricorn: constellation and SIGN of the ZODIAC, the Goat. [10.18–19] 53

Celestial SPHERE: the surface of the spherical COSMOS.

Circle of the CELESTIAL SPHERE: intersection of a plane with the solid SPHERE, a circular disc. [2.13] 44

Circumference of CIRCLE *of the* CELESTIAL SPHERE: intersection of a plane with the surface of the celestial sphere. [2.7] 43

Corresponding arcs of PARALLEL CIRCLES: any number of arcs SIMILAR to one another cut off by a single pair of GREAT circles. (The Greek word so trans-

lated, ἀπεναντίον, is usually rendered 'opposite', which is inappropriate in this instance because it pertains only to pairs.) [66.8] 84

Cosmos: the entire universe, including the Sun, Moon, Earth, planets, and fixed stars. In the introduction to the *Phaenomena*, it is argued and then postulated to have a spherical shape. [4.26] 45

Diopter: ancient THEODOLITE used both for astronomical and surveying purposes. Construction and use of such devices were the subjects of the *De dioptra* (*On the diopter*) by Heron of Alexandria. (For, extracts from this treatise, together with a picture of a reconstructed diopter, cf. Cohen and Drabkin 1948, 139, 336–342.) [10.16] 52

Earth: the centre of the CELESTIAL SPHERE. [4.2] 44

Ecliptic: a great CIRCLE of the CELESTIAL SPHERE skewed both to the equator and horizon and rotating with the CELESTIAL SPHERE around its AXIS. It is the apparent annual path of the centre of the Sun and the central curve of the ZODIAC. See pp. 27–31, above. [4.22] 45

Equator or equinoctial circle: the GREAT CIRCLE of the CELESTIAL SPHERE that is the largest of the PARALLEL CIRCLEs on which the FIXED stars move. [4.15] 45

Ever-invisible circle: PARALLEL CIRCLE of the CELESTIAL SPHERE, always below the observer's HORIZON. [12.25] 55

Ever-visible circle: PARALLEL CIRCLE of the CELESTIAL SPHERE, always above the observer's HORIZON. [2.19] 44

Fixed stars: non-wandering stars. These stars were said to be fixed to the surface of the CELESTIAL SPHERE because, unlike the planets (in Greek: wanderers), the only motion they were thought to exhibit at Euclid's time was that of the daily rotation of the COSMOS. [2.1] 43

Following arcs of the ECLIPTIC: arcs that come after a given arc in the direction of the Sun's annual motion around the ECLIPTIC, arcs that rise above the HORIZON or set below it after the given arc. [44.13] 74

Great CIRCLE *of the* CELESTIAL SPHERE: intersection of this sphere with a plane that passes through its centre. [6.24] 47

Horizon: GREAT CIRCLE of the CELESTIAL SPHERE dependent on the terrestrial location of the observer. This circle divides what is *theoretically* not obscured from view by the Earth from what is obscured, since the Earth is treated as a point at the centre of the COSMOS. (Neugebauer [1975, 756] calls the tenet that the Earth is at the centre of the cosmos the most important physical assumption in the subject of spherical astronomy.) Unlike the ECLIPTIC and other GREAT CIRCLES, the horizon does not rotate but, like the observer whose location defines it, stays still. The horizon so understood is not the observer's real horizon, which is defined by local geography and other factors. [2.23] 46

Interchange of arc on the CELESTIAL SPHERE: Neugebauer's more literal translation of the Greek term for PASSAGE ACROSS. [10.3] 47n26

Leo: constellation and SIGN of the ZODIAC, the Lion. [12.1] 53

Libra: constellation and SIGN of the ZODIAC, the Scales. [6.25] 47

Longer (shorter) than SIMILAR arc on CELESTIAL SPHERE: parallel arc subtending a larger (smaller) angle at the AXIS, therefore crossing the MERIDIAN in a longer (shorter) time. [48a.4 (66b.8)] 74 (88)

Meridian circle: a GREAT CIRCLE of the CELESTIAL SPHERE dependent on the terrestrial location of the observer, passing through the POLEs of the EQUATOR and meeting the observer's HORIZON at right angles. Like the HORIZON, it does not rotate. [6.19] 46

North: literally, (the regions) towards URSA MAJOR and URSA MINOR; that is, for Greeks and other observers in the Earth's northern hemisphere, towards the VISIBLE POLE of the EQUATOR. [4.18] 45

Opposite arcs: arcs that are diametrically opposite point by point. See ALTERNATE. [36.7] 69

Parallax: the difference in apparent position of celestial body on the CELESTIAL SPHERE resulting from difference in terrestrial location of observer. The second last sentence of the proof of Proposition 1 asserts the absence of parallax for the FIXED STARS.

Parallel circles: intersections with the CELESTIAL SPHERE of planes that are parallel in the ordinary sense and perpendicular to the AXIS of the celestial sphere. [2.15] 45

Passage across a hemisphere of the CELESTIAL SPHERE: the motion of an arc on the celestial sphere from the time the leading end enters the hemisphere until the trailing end leaves it. The term extends the obvious meaning for a point to an arc. (Neugebauer devotes a section [1975, 759–760] to this term under the label 'interchange' of hemispheres.) [10.3] 47

Phaenomena (Phenomena): literally, what appears or is observed; technically, celestial events of interest to astronomers; the name of number of treatises. [1.1] 43

Pole of a CIRCLE *of the* CELESTIAL SPHERE: each of the two points of intersection with surface of the celestial sphere of the perpendicular to the circle erected at its centre. Also the VISIBLE one of the two poles. [2.16] 44

Postulate: as verb—to introduce statements upon which the remainder of the work is based. See pp. 8–13. [2.8] 44

Rising part of the CELESTIAL SPHERE: any part's passing from below the HORIZON to above it. [2.4] 43

Rising time of an arc on the CELESTIAL SPHERE: the duration of the motion of an arc from the time its leading point is RISING until the trailing point is RISING. [56.5] 78

Setting part of the CELESTIAL SPHERE: any part's passing from above the HORIZON to below the horizon. [2.5] 44

Signs of ZODIAC: the twelve equal parts into which the band of the ZODIAC and, hence, the ECLIPTIC, are divided. The name of each sign is taken from a constellation approximately in its portion of the zodiacal band. The signs in the order of the Sun's annual motion, beginning with the vernal equinox, are: Aries, Taurus, Gemini, CANCER, LEO, Virgo, LIBRA, Scorpio, Sagittarius, CAPRICORN, AQUARIUS, and Pisces. According to Euclid, Aries begins at the vernal equinox. [6.21] 47

Similar arcs on the CELESTIAL SPHERE: PARALLEL arcs subtending equal angels at the AXIS, therefore crossing the MERIDIAN in equal times in spite of their having different lengths. [8.6] 47

South: literally, (the regions) towards (the Sun at) midday; that is, for Greeks and other observers in the Earth's northern hemisphere, south. [2.24] 44

Sphere: the solid shape POSTULATEd for the COSMOS. Its surface is the CELESTIAL SPHERE. [6.17] 46

Summer TROPIC: the northern TROPIC circle. [12.18] 55

Syzygy of two stars: literally, a yoking together; diametrical opposition of two stars on the CELESTIAL SPHERE. [6.26] 47

Theodolite: an optical instrument with a sighting piece (but not a telescope), mounted on a flat base and fitted with a level, used for astronomical observation and surveying.

Time of revolution of the COSMOS: the duration of the motion of a FIXED star from RISING to its next RISING or of its return to an arbitrary place with respect to the unmoving Earth. [8.29] 47

Tropic circle, one of the two PARALLEL CIRCLEs tangent to the ECLIPTIC that are described by the Sun on the days of the solstices. [6.21] 46

Turns of the ECLIPTIC: the apparent north-south wobble of the ECLIPTIC as more northerly or more southerly points of it rise or set during its rotation with the CELESTIAL SPHERE. [34.13] 68

Ursa Major and Ursa Minor: most northerly constellations, Great Bear and Lesser Bear. [2.11] 44

Visible events: cf. HORIZON. [2.19] 44

Visible POLE *of the* HORIZON: now called ZENITH, from an Arabic term. [12.18] 55

Winter TROPIC: the southern TROPIC circle described by the Sun at winter solstice. [12.27] 55

Zenith: see VISIBLE POLE of the HORIZON.

Zodiac: band around the CELESTIAL SPHERE centred on the ECLIPTIC, within which the Sun, Moon, and planets move. [4.22] 45

GREEK GLOSSARY OF SELECTED TECHNICAL TERMS AND PHRASES

For explanations, see English Glossary. We indiate the location (if any) in Menge 1916 where each term is first used, at the end of its entry in brackets in the format [page-number.line-number].

ἀνατολή rising [2.4]

ἀνατολῆς χρόνος rising time [56.5]

ἄξων axis [6.13]

ἀπενάντίον opposite (of arcs) [36.7]: corresponding [66.8]

ἀπλανῆ ἄστρα fixed stars [2.1]

ἀρτικὸς κύκλος arctic circle (astronomical) [2.21]

ἄρκτοι Ursae (Ursa Major and Ursa Minor) [2.11]

ἀσύμπτωτος asymptotic [22.11]

ἁφή point of contact [16.15]

βόρειος northern [44.13]

γάλακτος κύκλος galaxy, Milky Way [4.22]

γῆ Earth [4.2]

διάμετρος diameter [6.25]

δίοπτρα diopter [10.16]

δύσις setting [2.5]

ἐναλλάξ alternate (of arcs) [4.17]

ἐξαλλαγή interchange, passage across [10.3]

ζῳδιακὸς κύκλος ecliptic (circle) or zodiacal (band) [4.22]

ζῴδιον sign, (conventional twelfth) of zodiac or of ecliptic, or constellation associated with a sign [6.21]

θερινὸς (τροπικός) summer (northern) tropic [12.18]

ἰσημερινὸς (κύκλος) equator [4.15]

κόσμος cosmos [4.26]

κύκλος circle [2.13]

μεσημβρία south [2.24]

μεσημβρινὸς (κύκλος) meridian [6.19]

μετὰ + acc. following (of signs on the ecliptic) in the conventional order [44.13]

ὅμοιος similar [8.6]

ὀπτικά optics [2.8]

ὁρίζων (κύκλος) horizon [2.23]

παράλληλος parallel [2.15]

περιστροφῆς χρόνος time of revolution [8.29]

περιφέρεια circumference or arc [2.7]

πόλος pole [2.16]

πτῶσις case (in argument) requiring separate diagram [60.19]

συζυγία syzygy [6.26]

συναφή point of contact [40.7]

σφαῖρα sphere [6.17]

τμῆμα segment [4.3]

τροπή turn [34.13]

τροπικός tropic [6.21]

φαινόμενα phenomena (astronomical) [1.1]

φανερός visible [2.19]

χειμερινὸς (τροπικός) winter (southern) tropic [12.27]

BIBLIOGRAPHY

Archimedes. *Arenarius* (*Sand Reckoner*). For Greek text, see Heiberg 1913; for English translation, see Heath 1897.

Aristotle. *Analytica posteriora.* (*Posterior Analytics*). For Greek text and English translation, see Tredennick 1960.

Aujac, G. 1975. ed. and trans. *Géminos: Introduction aux phénomènes.* Paris.

Autolycus. *De sphaera quae movetur* (*On a Moving Sphere*). For Greek text and French translation, see Mogenet 1950; for English translation, see Bruin and Vondjidis 1971.

_____ *De ortibus et occasibus* (*On Risings and Settings*). For Greek text and French translation, see Mogenet 1950; for English translation, see Bruin and Vondjidis 1971.

Berggren, J. L. 1991. 'The Relation of Greek Spherics to Early Greek Astronomy.' See Bowen 1991b, 227–248.

Bowen, A. C. 1991a. 'Euclid's *Sectio canonis* and the History of Pythagoreanism.' See Bowen 1991b, 164–187.

_____ 1991b. ed. *Science and Philosophy in Classical Greece.* New York/ London.

Bruin, E. and Vondjidis, A. 1971. *The Books of Autolykos.* Beirut.

Cohen, M. R. and Drabkin, I. E. 1948. *A Source Book in Greek Science.* Cambridge, MA.

Cornford, F. M. 1941. trans. *The Republic of Plato.* Oxford.

De Falco, V. and Krause, M. 1966. ed. and trans. *Hypsicles: Die Aufgangzeite der Gestirne.* Abh. der Akad. der Wissen. in Göttingen philolog.-historische Klasse (3) 62.

Euclid. *Elementa* (*Elements*). For Greek text, see Heiberg and Stamatis 1969–1977; for English translation, see Heath 1925.

_____ *Phaenomena.* For Greek text, see Menge 1916.

———— *Sectio canonis*. For Greek text, see Menge 1916; for English translation of the introduction, see Bowen 1991a.

Friedlein, G. 1873. ed. *Procli diadochi in primum Euclidis elementorum librum commentarii*. Leipzig.

Galen. *De placitis Hippocratis et Platonis* (*On the Doctrines of Hippocrates and Plato*). For Greek text, see Müller 1874.

Geminus. *Introductio astronomiae* (*Introduction to Astronomy*). For Greek text and German translation, see Manitius 1898; for Greek text and French translation, see Aujac 1975.

Goldstein, B. R. and Bowen, A. C. 1983. 'A New View of Early Greek Astronomy' *Isis* 74:330–340.

Heath, T. L. 1897. trans. *The Works of Archimedes*. Cambridge. Repr. with the *The Method of Archimedes*, New York: Dover Publications, [n.d.].

———— 1913. *Aristarchus of Samos: The Ancient Copernicus*. Oxford. Repr. New York: Dover Publications, 1981.

———— 1921. *A History of Greek Mathematics*. 2 vols. Oxford.

———— 1925. trans. *The Thirteen Books of Euclid's* Elements. 3 vols. Cambridge. Repr. New York: Dover Publications, 1956.

———— 1932. *Greek Astronomy*. London/Toronto. Repr. New York: Dover Publications, 1991.

Heiberg, J. L. 1882. *Litterargeschichtliche Studien über Euklid*. Leipzig.

———— 1883–1916. ed. *Euclidis opera omnia*. 8 vols. Leipzig.

———— 1898–1903. ed. *Claudii Ptolemaei opera quae exstant omnia: Syntaxis mathematica*. 2 vols. Leipzig.

———— 1913. ed. *Archimedis opera cum commentariis Eutociis*. ii. Leipzig.

Heiberg, J. L. and Stamatis, E. S. 1969–1977. ed. *Euclidis elementa cum scholiis et appendicibus*. 5 vols. Leipzig.

Heron of Alexandria. *De dioptra* (*On the diopter*). See Cohen and Drabkin 1948.

Hipparchus. *In Arati et Eudoxi phaenomena commentaria*. (*Commentary on the* Phaenomena *of Aratus and Eudoxus*). See Manitius 1894.

Hultsch, F. 1875–1878. ed. *Pappi Alexandrini collectionis quae supersunt*. 3 vols. Berlin.

———— 1886. 'Autolycus und Euclid' *Berichte über die Verhandlungen der Kgl. Sächs. Gesell. der Wissenschaften zu Leipzig*, philolog.-historische Klasse 38:128–155.

Hunt, J. 1707. *Theodosii sphaericorum libri tres*. Oxford.

Hypsicles. *Anaphoricus* (*On Risings*). For Greek text and German translation, see De Falco and Krause 1966.

Jan, K. von. 1895. ed. *Musici scriptores graeci.* Leipzig.

Kuhn, T. S. 1957. *The Copernican Revolution.* Cambridge, MA.

Manitius, C. 1894. ed. *Hipparchi in Arati et Eudoxi phaenomena commentariorum libri tres.* Leipzig.

_____ 1898. ed. *Gemini elementa astronomiae.* Leipzig.

McKirahan, R. D. 1992. 'Euclid's Principles and Aristotle's: A Study in the Interaction between Mathematics and Philosophy.' A paper presented at the International Symposium on Early Greek Mathematics, August 17–21, 1992, in Athens, Greece.

Menge, H. 1916. ed. *Euclidis phaenomena et scripta musica.* See Heiberg 1883–1916, viii.

Mogenet, J. 1950. *Autolycus de Pitane: Histoire du texte, suivie de l'édition critique des traités* De la sphère en mouvement *et* Des levers et couchers. Université de Louvain, recueil de travaux d'histoire et de philologie, 3e sér., fasc. 37. Louvain.

_____ 1985. *Le "Grand Commentaire" de Théon d'Alexandrie aux tables faciles de Ptolémée, livre I: Histoire du texte, édition critique, traduction, revues et complétées par Anne Tihon, commentaire par Anne Tihon.* Studi e Testi No. 315. Vatican.

Morrow, G. R. 1970. trans. *Proclus: A Commentary on the First Book of Euclid's Elements.* Princeton.

Müller, I. 1874. ed. *De placitis Hippocratis et Platonis.* Leipzig.

Neugebauer, O. 1955. *Astronomical Cuneiform Texts.* 3 vols. London.

_____ 1975. *A History of Ancient Mathematical Astronomy.* Berlin/Heidelberg/New York.

Nokk, A. 1850. trans. *Euklid's Phaenomene.* Freiburg.

Pappus. *Collectio* (*Collection*). For Greek text, see Hultsch 1875–1878; for French translation, see ver Eecke 1933.

Proclus. *In primum Euclidis elementorum librum commentarii* (*Commentary on the First Book of Euclid's* Elements). For Greek text, see Friedlein 1873; for English translation, see Morrow 1970.

Ptolemy. *Syntaxis mathematica* or *Almagestum* (*Almagest*). For Greek text, see Heiberg 1898–1903; for English translation, see Toomer 1984.

Theodosius. *Sphaerica* (*Spherics*). For Greek text, see Hunt 1707; for French translation, see ver Eecke 1927.

Theon of Alexandria. *Commentaria in Ptolemaei canones* (*Commentary on Ptolemy's Handy Tables*). For Greek text and French translation, see Tihon 1978, 1991 and Mogenet 1985.

Tihon, A. 1978. *Le "Petit commentaire" de Théon d'Alexandrie aux tables faciles de Ptolémée: Histoire du texte, édition critique, traduction.* Studi e Testi No. 282. Vatican.

_____1991. *Le "Grand Commentaire" de Théon d'Alexandrie aux tables faciles de Ptolémée, livres II et III: Edition critique, traduction, commentaire.* Studi e Testi No. 340. Vatican.

Toomer, G. J. 1984. trans. *Ptolemy's* Almagest. London.

Tredennick, H. 1960. trans. *Aristotle: Posterior Analytics.* Loeb Classical Library. London.

ver Eecke, P. 1927. trans. *Les sphériques de Théodose de Tripoli.* Bruges.

_____1933. trans. *Pappus d'Alexandrie, La collection mathématique.* 2 vols. Paris/Bruges.

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[1] See Index of Subjects: *s.v.* semicircles, asymptotic.

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The book contains a translation and study of Euclid's *Phaenomena*, a work which once formed part of the mathematical training of astronomers from Central Asia to Western Europe. Included is an introduction that sets Euclid's geometry of the celestial sphere, and its application to the astronomy of his day, into its historical context for readers not already familiar with it. So no knowledge of astronomy or advanced mathematics is necessary for an understanding of the work. The book shows mathematical astronomy shortly before the invention of trigonometry, which allowed the calculation of exact results and the subsequent composition of Ptolemy's *Almagest*.

The *Phaenomena* itself begins with an introduction (possibly not by Euclid) followed by eighteen propositions set out in geometrical style about how arcs of the zodiacal circle move across the sky. The astronomical application is to the small arc of that circle occupied by the Sun, but the Sun is not mentioned. This work and the (roughly) contemporaneous treatises of Autolycus and Aristarchos form a corpus of the oldest extant works on mathematical astronomy. Together with Euclid's *Optics* one has the beginnings of the history of science as an application of mathematics.

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